Buildings and Classical Groups

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Introduction

This book describes the structure of *the classical groups*, meaning general linear groups, symplectic groups, and orthogonal groups, both over general fields and in finer detail over p-adic fields. To this end, half of the text is a systematic development of the theory of buildings and BN-pairs, both spherical and affine, while the other half is illustration by and application to the classical groups.

The viewpoint is that buildings are the fundamental objects, used to study groups which act upon them. Thus, to study a group, one discovers or constructs a building naturally associated to it, on which the group acts nicely.

This discussion is intended to be intelligible after completion of a basic graduate course in algebra, so there are accounts of the necessary facts about geometric algebra, reflection groups, p-adic numbers (and other discrete valuation rings), and simplicial complexes and their geometric realizations.

It is worth noting that it is the *building-theoretic* aspect, not the *algebraic group* aspect, which determines the nature of the basic *representation theory* of p-adic reductive groups.

One important source of information for this and related material is the monumental treatise of Bruhat-Tits, which appeared in several parts, widely separated in time. This treatise concerned mostly application of the theory of affine buildings to p-adic groups of the theory of affine buildings. One of the basic points made, and an idea pervasive in the work, is that buildings can be attached in an intrinsic manner to all p-adic reductive groups. But this point is difficult to appreciate, making this source not congenial to beginners, presuming as it does that the reader knows a great deal about algebraic groups, and has a firm grasp of root systems and reflection groups, having presumably worked all the exercises in Bourbaki's Lie theory chapters IV,V,VI.

In contrast, it is this author's opinion that, rather than being corollaries of the theory of algebraic groups, *the mechanism by which a suitable action of a group upon a building illuminates the group structure is a fundamental thing itself.*

Still, much of the material of the present monograph can be found in, or inferred from, the following items:

F. Bruhat and J. Tits, *Groupes Reductifs sur un Corps Local, I: Donnees radicielles valuees*, Publ. Math. I.H.E.S. 41 (1972), pp. 5-252.

F. Bruhat and J. Tits, *Groupes Reductifs sur un Corps Local, II: Schemas en groups, existence d'une donnee radicielle valuee*', *ibid* 60 (1984), pp. 5-184.

F. Bruhat and J. Tits, *Groupes Reductifs sur un Corps Local, III: Complements et applications a la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo 34 (1987), pp. 671-688.

F. Bruhat and J. Tits, *Schemas en groupes et immeubles des groupes classiques sur un corps local*, Bull. Soc. Math. Fr. 112 (1984), pp. 259-301.

I have benefited from the quite readable

J. Humphreys, *Reflection Groups and Coxeter Groups*, Camb. Univ. Press, 1990.

K. Brown, *Buildings*, Springer-Verlag, New York, 1989.

M. Ronan, *Lectures on Buildings*, Academic Press, 1989.

Even though I do not refer to it in the text, I have given as full a bibliography as I can. Due to my own motivations for studying buildings, the bibliography also includes the representation theory of p-adic reductive groups, especially items which illustrate the use of the finer structure of p-adic reductive groups discernible via building-theory.

By 1977, after the first of the Bruhat-Tits papers most of the issues seem to have been viewed as 'settled in principle'. For contrast, one might see some papers of Hijikata which appeared during that period, in which he studied p-adic reductive groups both in a classical style and also in a style assimilating the Iwahori-Matsumoto result:

H. Hijikata, *Maximal compact subgroups of some p-adic classical groups*, mimeographed notes, Yale University, 1964.

H. Hijikata, *On arithmetic of p-adic Steinberg groups*, mimeographed notes, Yale University, 1964.

H. Hijikata, *On the structure of semi-simple algebraic groups over valuation fields, I*, Japan, J. Math. (1975), vol. 1 no. 1, pp. 225-300.

The third of these papers contains some very illuminating remarks about the state of the literature at that time.

Having made these acknowledgements, I will simply try to tell a coherent story.

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In rough geometric terms, a Coxeter group is one generated by *reflections*. Coxeter groups are very special among groups, but are also unusually important, arising as crucial auxiliary objects in so many different circumstances.

For example, symmetric groups (that is, full permutation groups) are Coxeter groups. and already illustrate the point that some of their properties are *best* understood by making use of the fact that they are Coxeter groups.

What we do here is the indispensable minimum, and is completely standard.

1.1 Words, lengths, presentations of groups

This little section recalls some standard and elementary concepts from general group theory.

Let G be a group with generators S. The **length** $\ell(g) = \ell_S(g)$ of an element g of G with respect to the generating set S is the least integer n so that g has an expression

$$g = s_1 \ldots s_n$$

with each $s_i \in S$. Any expression

$$g = s_1 \ldots s_n$$

with $n = \ell(g)$ is **reduced**. These expressions in terms of generators are **words** in the generators.

Let F be a free non-abelian group on generators $[s]$ for s in a set S. Thus, F consists of all *words*

$$[s_1]^{m_1} \ldots [s_n]^{m_n}$$

where the m_i are integers and the s_i are in S. Let X be a set of 'expressions' of the form $s_1^{m_1} \ldots s_n^{m_n}$ with all s_i in S. We wish to form the *largest quotient* group G of the free group F in which the *image* of $[s_1]^{m_1} \ldots [s_n]^{m_n}$ is 1 whenever $[s_1]^{m_1} \ldots [s_n]^{m_n}$ is in X. As should be expected, this quotient is obtained by taking the quotient of F by the smallest normal subgroup containing all words $[s_1]^{m_1} \ldots [s_n]^{m_n}$ in X. By an abuse of notation, one says that the group G is **generated by S with presentation**

$$\{s_1 \ldots s_n = 1 : \forall s_1^{m_1} \ldots s_n^{m_n} \in X\}$$

Of course, in general it is not possible to tell much about a group from a presentation of it. In this context, we should feel fortunate that we *can* so successfully study Coxeter groups, as follows.

1.2 Coxeter groups, systems, diagrams

This section just gives the basic definitions. Even the most fundamental facts will take a little time to verify, which we will do in the following sections.

Fix a set S, and let

$$m : S \times S \to \{1, 2, 3, \ldots, \infty\}$$

be a function so that $m(s,s) = 1$ for all $s \in S$ and so that $m(s,t) = m(t,s)$ for all $s,t \in S$. For brevity, we may write

$$m_{st} = m(s,t)$$

A **Coxeter system** is a pair (W,S) where S is a set of generators for a group W, and W has presentation

$$s^2 = 1 \ \ \forall s \in S$$

$$(st)^{m(s,t)} = 1 \ \ \forall s,t \in S$$

By convention, $m(s,t) = \infty$ means that no relation is imposed. Note that if $m(s,t) = 2$ then $st = ts$, since $s^2 = 1$ and $t^2 = 1$.

We may refer to the function m as giving **Coxeter data**.

It is an abuse of language to then say that W is a **Coxeter group**, since there are several reasons for keeping track of the choice of generating set S. Indeed, the usual interest in a group's being a 'Coxeter group' resides in reference to the set S.

A **dihedral group** is a Coxeter group with just two generators. At many points in the discussion below, issue are reduced to the analogues for *dihedral groups*, rendering computation feasible.

A **Coxeter diagram** is a schematic device often convenient to keep track of the numbers $m(s,t)$ which describe a Coxeter system (W,S): for each $s \in S$ we make a 'dot', connect the s-dot and t-dot by a line if $2 < m(s,t)$, and label this connecting line by $m(s,t)$ (if $m(s,t) > 2$). (Thus, a Coxeter diagram is a one-dimensional complex with vertices in bijection with the set S, etc.) When $m(s,t) = 3$ we may omit the label on the line segment connecting the s-dot and t-dot. The reason for this is that it turns out that (for $m(s,t) > 2$) the most common value of $m(s,t)$ is 3. And keep in mind that if $m(s,t) = 2$ then $st = ts$.

A Coxeter diagram is **connected** if, for all $s, t \in S$, there is a sequence

$$s = s_1, s_2, s_3, \ldots, s_n = t$$

so that $m(s_i, s_{i+1}) \geq 3$. That is, the diagram is connected if and only if it is connected as a one-dimensional simplicial complex.

Alternatively, we may say that a Coxeter system is **indecomposable** or **irreducible** if it is *connected* in this sense.

1.3 Linear representation, reflections, roots

Let (W, S) be a Coxeter system. One primitive goal is that of showing that the elements of S and their pairwise products truly have the orders that they would appear to have from the presentation of the group W. That is, the generators should have order 2 (that is, not 1), and a product st should have order $m(s,t)$ (rather than some *proper* divisor of $m(s,t)$). In the course of proving this we introduce Tits' linear representation of a Coxeter group defined by mapping the involutive generators S of W to suitable *reflections* in a real vectorspace. This representation is sometimes called the *geometric realization* of W.

Only after we have verified that the linear representation is indeed a group homomorphism can we define the *roots*.

Let $V = V_S$ be a real vectorspace with basis consisting of elements e_s for $s \in S$. Define a symmetric bilinear form $\langle , \rangle$ on V by

$$\langle e_s, e_t \rangle = -\cos(\pi / m_{st})$$

(For $m_{st} = \infty$, take $\langle e_s, e_t \rangle = -1$.) This is the **Coxeter form**.

Suppose that S is finite with cardinality n and that we have ordered S as $s_1, \ldots, s_n$. Then the **Coxeter matrix** associated to a Coxeter system (W, S) is the $n \times n$ matrix indexed by pairs of elements of S, with off-diagonal entries

$$B_{ij} = \frac{1}{2} \langle e_{s_i}, e_{s_j} \rangle$$

for $i \neq j$ and with diagonal entries 1.

Let G be the group of isometries of this bilinear form:

$$G = \{g \in GL(V) : \langle gx, gy \rangle = \langle x, y \rangle \ \ \forall x, y \in V\}$$

where $GL(V)$ is the group of $\mathbb{R}$-linear automorphisms of V. We may refer to G as the *orthogonal group of the form* $\langle , \rangle$ even though we certainly do not preclude the possibility that the form may be degenerate. For $s \in S$ define a **reflection** σ_s on V by

$$\sigma_s v = v - 2\langle v, e_s \rangle e_s$$

A direct computation shows that *these reflections lie in the orthogonal group* G.

Let Γ be the subgroup of the orthogonal group G generated by the reflections σ_s. We eventually want to see that the map

$$s_1 \ldots s_n \to \sigma_{s_1} \ldots \sigma_{s_n}$$

gives rise to a *group isomorphism*

$$W \to \Gamma$$

Knowing that this is an isomorphism is essential for the continuation. It is certainly not *a priori* clear that this map is even well-defined, since at the present point we do not know that the generators really are of order 2, not that products st really are of order $m(s,t)$, only that these orders *divide* 2 and $m(s,t)$, respectively.

One first step in proving this isomorphism is:

Lemma: *Each $s \in S$ is of order 2 in W.*

Proof: We make a group homomorphism ϵ from the free group F on generators S to $\{1,-1\}$ by $\epsilon(s) = -1$ (with the usual abuse of notation). Since ϵ vanishes on st for all $s, t \in S$, ϵ is compatible with the defining relations for the group and induces a group homomorphism $W \to \{1,-1\}$ with $\epsilon = -1$ on S. Thus, the generators S truly are of order 2. ♣

Next, to see that $s \to \sigma_s$ gives rise to a *group homomorphism* $W \to \Gamma$, we need to check that

$$(\sigma_s \sigma_t)^{m(s,t)} = 1$$

Fix $s \neq t \in S$, put $m = m_{st}$, and let

$$\lambda = -\cos(\pi/m)$$

The Coxeter form restricted to

$$U = \mathbb{R}e_s + \mathbb{R}e_t \subset V$$

has the matrix

$$\begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

which is positive definite if $m < \infty$ and (hence) $|\lambda| < 1$.

With respect to the ordered basis e_s, e_t, the reflections σ_s, σ_t restricted to U have matrices (respectively)

$$\begin{pmatrix} -1 & -2\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2\cos\frac{\pi}{m} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2\lambda & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2\cos\frac{\pi}{m} & -1 \end{pmatrix}$$

Thus, $\sigma_s \sigma_t$ restricted to U has matrix

$$\begin{pmatrix} -1 & -2\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2\lambda & -1 \end{pmatrix} = \begin{pmatrix} -1+4\lambda^2 & 2\lambda \\ -2\lambda & -1 \end{pmatrix}$$

One computes that for $m < \infty$ the eigenvalues of $\sigma_s\sigma_t$ restricted to U are $e^{\pm 2\pi i/m}$.

When $m = \infty$, $\lambda = -1$, and $\sigma_s\sigma_t$ restricted to U has matrix $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ which has Jordan form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so has infinite order, as desired.

Now if $m < \infty$, the subspace U is a *non-degenerate quadratic space*, in the sense of *geometric algebra.* (7.2) In that case, V is an *orthogonal* sum

$$V = U \oplus U^\perp$$

Since both σ_s and σ_t act trivially on $U^\perp$, then the order of $\sigma_s\sigma_t$ is exactly the order of this product restricted to U, which we have computed already. If $m = \infty$, then since the restriction of $\sigma_s\sigma_t$ is of infinite order, so must be the product $\sigma_s\sigma_t$.

Thus, so far, we have shown that the group homomorphism ϕ from the free group F on generators $[s]$ for $s \in S$ to Γ, defined by

$$[s_1]\ldots[s_n] \to \sigma_{s_1}\ldots\sigma_{s_n}$$

has in its kernel all expressions

$$[s]^2\ , \quad ([s][t])^{m(s,t)}$$

Thus, since the Coxeter group is defined to be the largest quotient of F in which such elements are mapped to the identity, we see that ϕ does indeed factor through W (although we do not yet have injectivity).

In summary, so far we have proven

Proposition: The linear representation of W described by

$$s_1\ldots s_n \to \sigma_{s_1}\ldots\sigma_{s_n}$$

is a group homomorphism, and the products $st \in W$ do indeed have orders $m(s,t)$. ♣

It still remains to prove *injectivity* of this map (below).

1.4 Roots and the length function

The notion of *root* discussed here is yet another crucial yet slightly obscure technical item. This notion itself, or analogues of it, comes up in many subjects.

For brevity, write simply

$$wv = \sigma_w v$$

for $v \in V$ and $w \in W$. That is, we identify W with its image under Tits' linear representation.

The set of **roots** of (W, S) is

$$\Phi = \{we_s : w \in W,\ s \in S\}$$

Note that all the vectors in Φ are of length 1, since the image Γ of W in $GL(V)$ lies in the orthogonal group of the Coxeter form, and the lengths of all the vectors e_s are 1 by definition of the Coxeter form. Since $se_s = -e_s$, we find

$$\Phi = -\Phi$$

For $\beta \in \Phi$, we can express β uniquely as

$$\beta = \sum_s c_s \, e_s$$

in terms of the basis e_s, with coefficients $c_s \in \mathbb{R}$. Say that a root $\beta \in \Phi$ is **positive** if for all $s \in S$ we have $c_s \geq 0$. We write this as $\beta \geq 0$. Say that a root $\beta \in \Phi$ is **negative** if for all $s \in S$ we have $c_s \leq 0$. We write this as $\beta \leq 0$. Let Φ^+ be the positive roots, and let Φ^- be the negative roots.

Lemma:

- $\ell(w) = \ell(w^{-1})$
- $\ell(ww') \leq \ell(w) + \ell(w')$
- $\ell(ww') \geq \ell(w) - \ell(w')$
- $\ell(w) - 1 \leq \ell(sw) \leq \ell(w) + 1$
- $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$

Proof is easy. ♣

Recall that we have defined

$$\epsilon : W \to \{\pm 1\}$$

by $\epsilon(s) = -1$ for any $s \in S$.

Lemma: For $w \in W$ and $s \in S$,

$$\epsilon(w) = (-1)^{\ell(w)}$$

and

$$\ell(ws) = \ell(w) \pm 1 \quad \ell(sw) = \ell(w) \pm 1$$

Proof of Lemma: Let $w = s_1 \ldots s_n$ be a reduced expression for w. Thus, $n = \ell(w)$, and $\epsilon(w) = (-1)^n$. Since

$$\epsilon(sw) = \epsilon(s)\epsilon(w) = -\epsilon(w)$$

it must be that $\ell(sw) \neq \ell(w)$. From this the result follows immediately, as does the symmetrical assertion for ws. ♣

Theorem: For $w \in W$ and $s \in S$,

$$\text{if} \quad \ell(ws) > \ell(w) \ \text{ then } \ we_s > 0$$

$$\text{if} \quad \ell(ws) < \ell(w) \ \text{ then } \ we_s < 0$$

Corollary: The linear representation $W \to GL(V)$ given by

$$s_1 \ldots s_n \to \sigma_{s_1} \ldots \sigma_{s_n}$$

is *injective*.

Proof of corollary: If there were $w \in W$ so that $wv = v$ for all $v \in V$, then certainly $we_s = e_s > 0$ for all $s \in S$. This implies, by the theorem, that $\ell(ws) > \ell(w)$ for *all* $s \in S$. This implies that $w = 1$: otherwise let $s_1 \ldots s_n$ be a reduced expression for w and take $s = s_n$ to obtain $\ell(ws) < \ell(w)$, contradiction. ♣

Proof of theorem: The second assertion of the theorem follows from the first: if $\ell(ws) < \ell(w)$, then

$$\ell((ws)s) = \ell(w) > \ell(ws)$$

so $wse_s > 0$. Then

$$we_s = ws(-e_s) = -(wse_s) < 0$$

We prove the first assertion by induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$. If $\ell(w) > 0$ then take $t \in S$ so that $\ell(wt) = \ell(w) - 1$, e. g. , we could take $t = s_n$ for $w = s_1 \ldots s_n$ a reduced expression for w. Then $s \neq t$. Let $T = \{s, t\}$, and let W_T be the subgroup of W generated by T. Then W_T is a *dihedral group*, that is , a Coxeter group with just two generators. Let ℓ_T be the length function on W_T with respect to the set of generators T.

Consider expressions $w = xy$ with $y \in W_T$ and $x \in W$. Let

$$X = \{x \in W : x^{-1}w \in W_T \ \text{ and } \ \ell(w) = \ell(x) + \ell_T(x^{-1}w)\}$$

Certainly $w = w \cdot 1$, so $w \in X$, and $X \neq \emptyset$. Choose $x \in X$ of least length, and let $y = x^{-1}w \in W_T$. Then $w = xy$ and

$$\ell(w) \leq \ell(x) + \ell(y) \leq \ell(x) + \ell_T(y)$$

We claim that $wt \in X$. Indeed,

$$(wt)^{-1} \cdot w = t \in W_T$$

and

$$\ell(wt) + \ell_T(t) = \ell(w) - 1 + 1 = \ell(w)$$

as desired. Thus,

$$\ell(x) \leq \ell(wt) = \ell(w) - 1$$

We can now do induction on $\ell(w)$ We claim that $\ell(xs) > \ell(x)$. If not, then

$$\ell(w) \leq \ell(xs) + \ell((xs)^{-1}w) \leq \ell(xs) + \ell_T(sx^{-1}w) =$$

$$= \ell(x) - 1 + \ell_T(sx^{-1}w) \leq \ell(x) - 1 + \ell_T(x^{-1}w) + 1 =$$

$$= \ell(x) + \ell_T(x^{-1}w) = \ell(w)$$

Then we could conclude that

$$\ell(w) = \ell(xs) + \ell_T((xs)^{-1}w)$$

and that $xs \in X$, contradicting the assumed minimal length of x among elements of X. Thus, we conclude that $\ell(xs) > \ell(x)$. By induction on $\ell(w)$, $xe_s > 0$. Similarly, we conclude that $\ell(xt) > \ell(x)$ and $xe_t > 0$.

It remains to show that $ye_s > 0$, e. g. , to show that

$$ye_s = ae_s + be_t$$

with $a, b \geq 0$, since then

$$we_s = (xy)e_s = x(ye_s)$$

and we already know how x acts on e_s, e_t. This is a question referring only to the dihedral group (Coxeter group on two generators) W_T. First, we claim that $\ell_T(ys) \geq \ell_T(y)$. Otherwise,

$$\ell(ws) = \ell(xx^{-1}ws) \leq \ell(x) + \ell(x^{-1}ws) = \ell(x) + \ell(ys) \leq$$
$$\leq \ell(x) + \ell_T(ys) < \ell(x) + \ell_T(y) = \ell(w) < \ell(ws)$$

giving a contradiction. Thus, *any* reduced expression for y in W_T must end in t.

Now we claim that *any* element y of the dihedral group W_T *all* of whose reduced expressions are of the form

$$y = \ldots t$$

has the property that

$$ye_s = ae_s + be_t$$

with $a, b \geq 0$.

If $m(s,t) = \infty$, then $\langle e_s, e_t \rangle = -1$, and

$$te_s = e_s - 2(-1)e_t = e_s + 2e_t$$
$$(st)e_s = s(e_s + 2e_t) = (e_s + 2e_t) - 2[1 + 2(-1)]e_s = 3e_s + 2e_t$$

and so on. By induction,

$$(st)^n e_s = (1 + 2n)e_s + 2ne_t \quad \text{and} \quad t(st)^n e_s = (1 + 2n)e_s + (2n + 2)e_t$$

giving the desired positivity assertion.

Suppose now that $m(s,t) = m < \infty$. First, we note that $\ell_T(y) < m$, since the element of W_T with length m *can* be written as

$$(st)^{m/2} = (ts)^{m/2} \quad \text{or} \quad (ts)^{(m-1)/2}t = (st)^{(m-1)/2}s$$

depending on whether m is even or odd. Thus, keeping in mind that $(st)^m = 1$, we need only consider y of the form

$$(m \text{ even})\ (st)^k \quad \text{with} \quad k < m/2$$
$$(m \text{ odd})\ t(st)^k \quad \text{with} \quad k < (m-1)/2$$

Completion of this proof now can be accomplished by direct computation. For brevity, let $\zeta = e^{2\pi i/m}$, a m^{th} root of unity. One computes that $\zeta^{\pm 1}e_s + e_t$ is a $\zeta^{\pm 1}$ eigenvector for st. Thus,

$$(\zeta - \zeta^{-1})e_s = (\zeta e_s + e_t) - (\zeta^{-1}e_s + e_t)$$

expresses e_s as a linear combination of eigenvectors, and

$$(\zeta - \zeta^{-1})(st)^k e_s = (\zeta^{2k+1}e_s + \zeta^{2k}e_t) - (\zeta^{-2k-1}e_s + \zeta^{-2k}e_t)$$

From this,

$$(st)^k e_s = \frac{\sin \frac{(2k+1)2\pi}{2m}}{\sin \frac{2\pi}{2m}} e_s + \frac{\sin \frac{(2k)2\pi}{2m}}{\sin \frac{2\pi}{2m}} e_t$$

We leave the rest to the reader. ♣

1.5 More on roots and lengths

The previous section was really just preparation. Now we can proceed to the heart of the matter.

Corollary: We have

$$\Phi = \Phi^+ \sqcup \Phi^-$$

Proof: First, note that this assertion is not *a priori* clear. Recall that Φ is the collection of all images we_s. Given $w \in W$ and $s \in S$, either $\ell(ws) > \ell(w)$, in which case (by the theorem) $we_s \in \Phi^+$, or $\ell(ws) < \ell(w)$, in which case (by the theorem) $we_s \in \Phi^-$. ♣

Corollary: The reflection $s \in GL(V)$ has the effect $se_s = -e_s$, and s *merely permutes the other positive roots.* More generally,

$$\ell(w) = \mathrm{card}\{\beta \in \Phi^+ : w\beta < 0\}$$

Proof: From the definition of the reflection (attached to) s, $se_s = -e_s$. Now let β be a positive root other than e_s. Since $\langle \beta, \beta \rangle = 1 = \langle e_s, e_s \rangle$, β and e_s are not collinear. Thus, in writing

$$\beta = \sum c_s e_s$$

with all $c_s \geq 0$, some $c_t > 0$ for $s \neq t \in S$. Then $s\beta - \beta \in \mathbb{R}e_s$ (from the definition of the action of the reflection s), so the coefficient of e_t in $s\beta$ is still $c_t > 0$. Therefore, $s\beta \notin \Phi^-$. The previous corollary then implies that $s\beta \in \Phi^+$. That is,

$$s(\Phi^+ - \{e_s\}) \subset \Phi^+ - \{e_s\}$$

Applying s again gives the equality asserted.

To prove the second assertion we make pointed use of the first. Let

$$\nu(w) = \mathrm{card}(\Phi^+ \cap w^{-1}\Phi^-)$$

be the number of positive roots sent to negative roots. The previous assertion shows that for $s \in S$ we do have $\nu(s) = \ell(s)$. Now do induction on length. It suffices to show that

$$we_s > 0 \Rightarrow \nu(ws) = \nu(w) + 1$$

and

$$we_s < 0 \Rightarrow \nu(ws) = \nu(w) - 1$$

If $we_s > 0$, then

$$\Phi^+ \cap (ws)^{-1}\Phi^- = s(\Phi^+ \cap w^{-1}\Phi^-) \sqcup \{e_s\}$$

where we use the first assertion to obtain the equality. This visibly has cardinality one greater than the cardinality of

$$\Phi^+ \cap w^{-1}\Phi^-$$

as desired. If $we_s < 0$, then

$$\Phi^+ \cap (ws)^{-1}\Phi^- = (s(\Phi^+ \cap w^{-1}\Phi^-) - \{e_s\}$$

so this set has cardinality one less than

$$\Phi^+ \cap w^{-1}\Phi^-$$

as desired. ♣

Corollary: If the Coxeter group W is *finite*, then there is a unique element w_o in W of maximal length, this maximal length is equal to the number of positive roots, and w_o maps every positive root to a negative root.

Proof: If there were two elements which mapped every positive root to a negative, then their product would send all positive to positive, so would have length 0. Thus, there is *at most* one element of W which sends all positive roots to negative.

Let w_o be a longest element in W. If $we_s < 0$ for all $s \in S$, then certainly $w\Phi^+ = \Phi^-$, since all positive roots are non-negative linear combinations of the e_s. If $we_s > 0$ for some $s \in S$, then (from above), $\ell(ws) > \ell(w)$, contradiction. ♣

1.6 Generalized reflections

This section extends our earlier discussion of Coxeter groups somewhat, mostly for the purpose of completing our discussion of *roots*.

For a root $\beta = we_s$ of a Coxeter group, we define the **associated reflection**

$$s_\beta v = v - 2\langle v, \beta\rangle \beta$$

Rewriting $\beta = we_s$, we see that

$$s_\beta v = v - 2\langle v, we_s\rangle we_s = v - 2\langle w^{-1}v, e_s\rangle we_s$$

(by the W-invariance which $\langle , \rangle$ has almost by definition)

$$= w(w^{-1}v - 2\langle w^{-1}v, e_s\rangle e_s) = wsw^{-1}v$$

That is, the 'generalized' reflection s_β is just a conjugate in W of one of the 'original' reflections s.

Lemma: The map $\beta \to s_\beta$ is a bijection from *positive* roots to reflections. We have $s_{-\beta} = s_\beta$.

Proof: The last assertion is easy to check. If $s_\beta = s_\gamma$ for two positive roots, then
$$-\beta = s_\beta(\beta) = s_\gamma(\beta) = \beta - 2\langle\beta,\gamma\rangle\gamma$$
which implies that $\beta = \langle\beta,\gamma\rangle\gamma$. Since both are unit vectors and are in Φ^+, we must have equality. ♣

Lemma: If α, β are roots and $\beta = w\alpha$ for some $w \in W$, then $ws_\alpha w^{-1} = s_\beta$. (The proof is direct computation, using the W-invariance of $\langle,\rangle$). ♣

Proposition: For $w \in W$ and $\alpha \in \Phi^+$, $\ell(ws_\alpha) > \ell(w)$ if and only if $w\alpha > 0$.

Proof: It suffices to prove the 'only if', since we can also consider the statement with w replaced by ws_α.

We do induction on the length of w. If $\ell(w) > 0$ then there is $s \in S$ so that $\ell(sw) < \ell(w)$. Then
$$\ell(sws_\alpha) \geq \ell(ws_\alpha) - 1 > \ell(w) - 1 = \ell(sw)$$
By induction on length, $(sw)\alpha > 0$. Suppose that $w\alpha < 0$. The only negative root made positive by s is $-e_s$, so necessarily $w\alpha = -e_s$. Then $sw\alpha = e_s$, and
$$(sw)s_\alpha(sw)^{-1} = s$$
by the previous lemma. Thus, $ws_\alpha = sw$. But this contradicts
$$\ell(ws_\alpha) > \ell(w) > \ell(sw)$$
Thus, we conclude that $w\alpha > 0$. ♣

1.7 Exchange Condition, Deletion Condition

The point of this section is to show that the assertion that (W, S) is a Coxeter system is equivalent to some other somewhat less combinatorial assertions, which lend themselves to a geometric reinterpretation. The execution and exploitation of this reinterpretation will occupy much of the remainder of the sequel.

One should note that in some sources Coxeter groups are *defined* by these other conditions. We do indeed prove the equivalence of these conditions: this is J. Tit's theorem proven just below.

The first of these alternative characterizations is the **Strong Exchange Condition:**

Theorem: Let $w = s_1 \ldots s_n$. If there is a (generalized) reflection t so that $\ell(wt) < \ell(w)$, then there is an index i so that
$$wt = s_1 \ldots \hat{s}_i \ldots s_n$$
(where the hat denotes omission). If the expression $w = s_1 \ldots s_n$ is *reduced*, then there is a unique such index.

Proof: Let $t = s_\alpha$ for some positive root α. Since $\ell(wt) < \ell(w)$ and $\alpha > 0$, from the previous section we conclude that $w\alpha < 0$. Thus, there is an index i so that $s_{i+1} \dots s_n \alpha > 0$ but $s_i s_{i+1} \dots s_n \alpha > 0$. Now the only positive root sent to its negative by s_i is e_{s_i}, so necessarily $s_{i+1} \dots s_n \alpha = e_{s_i}$. The lemma of the previous section then gives

$$(s_{i+1} \dots s_n) t (s_{i+1} \dots s_n)^{-1} = s_i$$

which can be rearranged to

$$wt = (s_1 \dots s_i)(s_{i+1} \dots s_n t) = (s_1 \dots s_i) s_i (s_n \dots s_{i+1})^{-1}$$

which yields the assertion.

Suppose that $n = \ell(w)$, and that s_i and s_j (with $i < j$) both could be 'deleted' in the above sense. From

$$s_1 \dots \hat{s}_i \dots s_n = wt = s_1 \dots \hat{s}_j \dots s_n$$

we cancel to obtain

$$s_{i+1} \dots s_j = s_i \dots s_{j-1}$$

That is, we have

$$s_i \dots s_j = s_{i+1} \dots s_{j-1}$$

so

$$w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_n$$

would be a shorter expression for w, contradiction. ♣

The following corollary is the **Deletion Condition:**

Corollary: If $w = s_1 \dots s_n$ with $n > \ell(w)$, then there are $i < j$ so that

$$w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_n$$

Indeed, a *reduced* expression for w may be obtained from this expression by deleting an *even* number of the s_i's.

Proof: First we claim that there is an index j (possibly $j = n$) so that

$$\ell(s_1 \dots s_j) < \ell(s_1 \dots s_{j-1})$$

Indeed, otherwise (by induction on j), we could prove that $\ell(w) = n$.

Then from

$$\ell((s_1 \dots s_j)s_j) = \ell(s_1 \dots s_{j-1}) > \ell(s_1 \dots s_j)$$

the Exchange Condition implies that there is an index $1 \le i \le j$ so that

$$s_1 \dots s_{j-1} = (s_1 \dots s_j) s_j = s_1 \dots \hat{s}_i \dots s_{j-1}$$

as desired. ♣

The next corollary illustrates the mechanism at work, and will have some use later.

Corollary: Given $w \in W$ and $s, t \in S$ with

$$\ell(sw) = \ell(w) + 1$$

$$\ell(wt) = \ell(w) + 1$$

either

$$\ell(swt) = \ell(w) + 2$$

or $swt = w$.

Proof: Let $w = s_1 \ldots s_n$ be a reduced expression for w. From the length hypotheses, $s_1 \ldots s_n t$ is a reduced expression for wt. By the Exchange Condition, either

$$\ell(s(wt)) = \ell(wt) + 1 = \ell(w) + 1 + 1 = \ell(w) + 2$$

or else we can exchange one of the letters in $s_1 \ldots s_n t$ for an s on the left end of the expression. The hypothesis $\ell(sw) > \ell(w)$ precludes exchange of one of the s_i for s, so the exchange must be for the final t:

$$ss_1 \ldots s_n = s_1 \ldots s_n t$$

That is, $sw = wt$, so $swt = (wt)t = w$ as claimed. ♣

Now we prove Tits' converse.

Theorem: Any group W generated by a set S with all $s \in S$ of order 2 and satisfying the Deletion Condition gives a Coxeter system (W, S).

Proof: We claim that all relations in W are 'derivable' from any relations of the special form $(st)^m = 1$ for $s, t \in S$. That is, we claim that all relations in W are derivable from the Coxeter-type relations among the generators (and from the relations $s_i^2 = 1$).

Given a relation $s_1 \ldots s_n = 1$ with all $s_i \in S$, we must show that this relation is implied by Coxeter-type relations. We do induction on n.

First, we claim that n must be *even* for there to be any such relation. To see this, 'define'

$$\epsilon(s_1 \ldots s_n) = (-1)^n$$

We will use the Deletion Condition to show that ϵ is a well-defined ± 1-valued function on W, from which it then will follow immediately that n must be *even* if such a relation holds. Indeed, if

$$s_1 \ldots s_m = t_1 \ldots t_n$$

with all $s_i, t_j \in S$ and with $m < n$, then $t_1 \ldots t_n$ is not reduced, and the Deletion Condition implies that there is a pair i, j of indices (with $i < j$) so that

$$t_1 \ldots t_n = t_1 \ldots \hat{t_i} \ldots \hat{t_j} \ldots t_n$$

Thus, the length of the word is decreased, but the *parity* of the length stays the same. Altogether, this gives the result.

If $n = 2$, the condition $s_1 s_2 = 1$ immediately gives $s_1 = s_2$, since always $s_i^2 = 1$. But then this is nothing but the assertion that $s_1^2 = 1$.

Before proceeding further, we make some general observations. For example, suppose that

$$s_1 \ldots s_n = 1$$

with $n = 2m$ and $m > 1$. Then we could infer that

$$s_1 s_2 \ldots s_{i-1} s_i = s_n s_{n-1} \ldots s_{i+1}$$

using only that all the elements s_j are of order 2, by *right* multiplying by $s_n s_{n-1} \ldots s_{i+2} s_{i+1}$. Further, by *left* multiplying the latter by $s_{i+1} s_{i+2} \ldots s_{n-1} s_n$ we could obtain

$$s_{i+1} \ldots s_n s_1 s_2 \ldots s_{i-1} s_i = 1$$

Thus, from a relation $s_1 \ldots s_n = 1$, (with $n = 2m$) we have the relation

$$s_1 \ldots s_{m+1} = s_n \ldots s_{m+2}$$

The length of the right-hand side is necessarily $\leq m-1$, so the left-hand side is surely not reduced. Thus, by the Deletion Condition there are $i < j \leq m+1$ so that

$$s_1 \ldots s_{m+1} = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_{m+1}$$

Doing some cancellation in the last equation, we have

$$s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$$

which (by right multiplication by $s_j \ldots s_{i+1}$) gives

$$s_i s_{i+1} \ldots s_{j-1} s_j s_{j-1} \ldots s_{i+2} s_{i+1} = 1$$

If we are *lucky* enough that the latter relation involves fewer than n reflections, then (by induction) it is derivable from the Coxeter-type relations, so the relation

$$s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$$

is so derivable. Then replace $s_{i+1} \ldots s_j$ by $s_i \ldots s_{j-1}$ in the original $s_1 \ldots s_n = 1$ and rewrite the latter as

$$1 = s_1 \ldots s_i (s_i \ldots s_{j-1}) s_{j+1} \ldots s_n = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_n$$

Again by induction, the relation

$$1 = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_n$$

is derivable from the Coxeter relations.

Therefore, in the *lucky* case, assuming the *truth* of $s_1 \ldots s_n = 1$, we know that the relations

$$s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$$
$$1 = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_n$$

are *derivable* from the Coxeter-type relations. Then we can derive $s_1 \ldots s_n = 1$ from these relations as follows:

$$s_1 \ldots s_n = s_1 \ldots s_i (s_{i+1} \ldots s_j) s_{j+1} \ldots s_n =$$
$$= s_1 \ldots s_i (s_i \ldots s_{j-1}) s_{j+1} \ldots s_n = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_n = 1$$

with all relations derivable from the Coxeter relations.

Now consider the *unlucky* possibility that

$$s_i s_{i+1} \ldots s_{j-1} s_j s_{j-1} \ldots s_{i+2} s_{i+1} = 1$$

still has n factors. Thus, even though we know this relation to hold (from the assumption that $s_1 \ldots s_n = 1$ is true), we cannot hope to invoke induction on length to say that we know that it is derivable from the Coxeter-type relations. This unlucky case occurs only if $i = 1$ and $j = m + 1$ and if the relation is

$$s_2 \ldots s_{m+1} = s_1 \ldots s_m$$

We could just rewrite this as

$$s_2 \ldots s_n s_1 = 1$$

and try the *lucky case* procedure as just above on this variant.

We would succeed in showing that *this* variant relation is derivable from the Coxeter ones unless we are doubly unlucky, in that we do not decrease the number of factors by using our first trick on the variant relation. This second failure will occur only if

$$s_3 \ldots s_{m+2} = s_2 \ldots s_{m+1}$$

With both failures, we now rather try to prove that the 'obstacle relation'

$$s_3 \ldots s_{m+2} = s_2 \ldots s_{m+1}$$

follows from the Coxeter relations. If we can show this, then we can substitute this relation into the original $s_1 \ldots s_n = 1$ and succeed. We can rewrite the obstacle relation as

$$s_3(s_2 s_3 \ldots s_{m+1}) s_{m+2} s_{m+1} \ldots s_4 = 1$$

Again the left-hand side has n factors, so we could try our first trick. We will succeed unless (as before)

$$s_2 \ldots s_{m+1} = s_3 s_2 s_3 \ldots s_m$$

Combining this with the relation

$$s_2 \ldots s_{m+1} = s_1 \ldots s_m$$

from above, we have $s_1 = s_3$.

That is, if $s_1 \neq s_3$ then the above scheme would work. Or, by cyclically permuting the relation $s_1 \ldots s_n = 1$ into the form

$$s_i s_{i+1} \ldots s_n s_1 s_2 \ldots s_i = 1$$

we can succeed if $s_2 \neq s_4$ or if $s_3 \neq s_5$, and so on. Thus, by induction again, we succeed unless

$$s_1 = s_3 = s_5 = \ldots \quad \text{and} \quad s_2 = s_4 = s_6 = \ldots$$

In the latter case, the original relation itself was actually

$$s_1 s_2 s_1 s_2 s_1 \ldots s_1 s_2 = 1$$

which is a Coxeter relation. ♣

1.8 The Bruhat order

The Bruhat ordering is a partial ordering on a Coxeter group which we will use in an essential way in the subsequent study of 'parabolic' subgroups of a Coxeter group.

(A subtler use, in case W is a Weyl group in a linear reductive p-adic (or Lie) group, is in description of the *topological relationships between the cells in a Bruhat decomposition*).

For purposes of this section, let T be the set of all ('generalized') reflections in a Coxeter group W (with generators S). That is, T includes not only the 'reflections" S, but also all *conjugates* in W of elements of S. For $v, w \in W$ write $v \to w$ if there is $t \in T$ so that $vt = w$ and $\ell(v) < \ell(w)$. Define the **Bruhat order** $<$ by saying that $v < w$ if there is a sequence

$$v = w_o \to w_1 \to w_2 \to \ldots \to w_n = w$$

This gives a partial ordering.

Remarks: It is not clear *a priori* that $v \to w$ implies that $\ell(v) = \ell(w) - 1$, since the definition of $v \to w$ does not require that $vs = w$ with $s \in S$, but only $vt = w$ with $t \in T$. Still, clearly $v \to w$ *does imply that the lengths of* v *and* w *are of opposite parity.*

Remarks: Superficially, it would appear that we could define another ordering by replacing $w't$ by tw' in the above. However, a moment's reflection indicates that allowing t to be in the collection T of *generalized* reflections, and not just in S, makes the 'left' and 'right' definitions equivalent. If, by contrast, we give the analogous definition with not T but S, then the distinction between vt and tv becomes significant. The latter ordering is sometimes called a **weak Bruhat order**.

Proposition: Let $v \leq w$ and take $s \in S$. Then either $vs \leq w$ or $vs \leq ws$ *or both.*

Proof: First consider $v \to w$ with $vt = w$ for $t \in T$ and $\ell(v) < \ell(w)$. If $s = t$, then $vs = w \leq w$ as desired.

Then suppose that $s \neq t$. If $\ell(vs) = \ell(v) - 1$, then $vs \to v \to w$, so we have $vs \leq w$. If $\ell(vs) = \ell(v) + 1$, then we claim that $vs < ws$. Let $t' = sts \in T$. We have $(vs)t' = ws$. Thus, by the definition of the Bruhat order, to prove $vs < ws$ it suffices to prove that $\ell(vs) < \ell(ws)$. Recall that $v \to w$ implies that the lengths have opposite parities. Thus, if we do not have $\ell(vs) < \ell(ws)$, then we can only have $\ell(vs) > \ell(ws)$. Take a reduced expression $v = s_1 \ldots s_n$. Still

$$vs = s_1 \ldots s_n s$$

is reduced, since

$$\ell(vs) > \ell(ws) \geq \ell(w) - 1 = \ell(v) + 1 - 1 = \ell(v)$$

implies that $\ell(vs) = \ell(v) + 1$. Then

$$\ell((vs)t') = \ell(ws) < \ell(vs)$$

implies, via the Strong Exchange Condition, that

$$vst' = s_1 \ldots \hat{s_i} \ldots s_n s$$

The omitted factor cannot be the last s, or else we would have

$$s_1 \ldots s_n st' = s_1 \ldots s_n$$

which would imply $s = t'$, that is , $s = sts$, that is , $s = t$. We supposed that this was not so. Thus, indeed,

$$ws = vst' = s_1 \ldots \hat{s_i} \ldots s_n s$$

and

$$w = s_1 \ldots \hat{s_i} \ldots s_n$$

which contradicts $\ell(v) < \ell(w)$.

More generally, suppose that

$$v = w_1 \to \ldots \to w_n = w$$

Already we have shown that either $vs \le w_2$ or $vs \le w_2 s$. In the former case, then we have (by transitivity)

$$vs \le w_2 \le w \Rightarrow vs \le w$$

In the latter case, by induction on n, we have

$$vs \le w_2 s \le ws \Rightarrow vs \le ws$$

This proves the proposition. ♣

Theorem: Let $w = s_1 \ldots s_n$ be a fixed reduced expression of $w \in W$. Then $v \le w$ if and only if v can be obtained as a subexpression of $s_1 \ldots s_n$, that is , if and only if v can be written in the form

$$v = s_{i_1} \ldots s_{i_m}$$

where

$$1 \le i_1 < i_2 < \ldots < i_m \le n$$

Proof: If $v \to w$ with $vt = w$, then since $\ell(v) < \ell(w)$ the Strong Exchange Condition can be applied to yield

$$v = wt = s_1 \ldots \hat{s_i} \ldots s_n$$

for some index I. If, further, $u \to v$ with $ut' = v$, then again the Strong Exchange Condition gives

$$u = vt' = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_n$$

or

$$u = vt' = s_1 \ldots \hat{s_j} \ldots \hat{s_i} \ldots s_n$$

for some other index j, depending upon whether $j > i$ or $j < i$. This trick can be continued, showing that $v \le w$ implies that v can be written as a subexpression of $s_1 \dots s_n$.

On the other hand, consider $v = s_{i_1} \dots s_{i_m}$. Certainly $\ell(v) \le \ell(w)$. Do induction on the length of w. If $i_m < n$, then apply the induction hypothesis to the necessarily reduced expression $s_1 \dots s_{n-1}$ to obtain

$$s_{i_1} \dots s_{i_m} \le s_1 \dots s_{n-1} = ws_n < w$$

If $i_m = n$, then, by induction,

$$s_{i_1} \dots s_{i_{m-1}} \le s_1 \dots s_{n-1}$$

We apply the previous proposition to obtain either

$$s_{i_1} \dots s_{i_m} \le s_1 \dots s_{n-1}$$

or

$$s_{i_1} \dots s_{i_m} \le s_1 \dots s_n = w$$

This proves the theorem. ♣

Corollary: For given $w \in W$ the set of elements of S occuring in any reduced expression for w depends only upon w, and not upon the particular reduced expression.

Proof: Let w be a counterexample of least length. Let $w = s_1 \dots s_m$ and $w = t_1 \dots t_n$ be two reduced expressions with all $s_i, t_j \in S$. Let I (resp. J) be the set of all s_i's (resp. , t_j's). The expression $v = s_2 \dots s_m$ is necessarily reduced, and, by the theorem, $v < w$. Since $\ell(v) < \ell(w)$, by induction the elements of S occuring in any reduced expression for v *is* well-defined, and equal to $\{s_2, \dots, s_m\}$. Also by the theorem, v can be written as a subexpression of $t_1 \dots t_n$, so has a reduced expression using elements of J. A similar discussion applies to $v' = s_1 \dots s_{m-1}$. Then we see that $I \subset J$. By symmetry, we have $I = J$, contradiction. ♣

Corollary: Let $v < w$ in W. Then there are elements $w_1, \dots, w_n$ in W so that $v = w_1 < \dots < w_n = w$ and $\ell(w_i) + 1 = \ell(w_{i+1})$ for all i.

Proof: Do induction on $\ell(v) + \ell(w)$. If this sum of lengths is 1, then $v = 1$ and $w \in S$.

Since $w \ne 1$, there is some $s \in S$ so that $\ell(ws) < \ell(w)$. Indeed, take $s = s_n$ for some reduced expression $w = s_1 \dots s_n$. The theorem implies that v is a subexpression

$$v = s_{i_1} \dots s_{i_m}$$

First consider the case that $v < vs$, that is , that $\ell(v) < \ell(vs)$. Then necessarily $i_m < n$. Then v is a subexpression of $ws < w$, and induction applies.

Second, consider the case that $v > vs$. (Note that v and vs always are comparable in the Bruhat order). Induction on the sum of the lengths gives a chain

$$vs = w_1 < \ldots w_m = w$$

where the lengths of successive elements differ by 1. Let i be the smallest index for which $w_i s < w_i$. Since $w_1 s = v > vs = w_1$ and $w_m s = ws < w = w_m$, such index does exist. Then we claim that $w_i = w_{i-1}s$. If not, we apply the Lemma below to

$$w_{i-1} < w_{i-1}s \neq w_i$$

to get $w_i < w_i s$, contrary to the defining property of i. Thus, indeed, $w_i = w_{i-1}s$.

On the other hand, for $1 \leq j < i$, we have $w_j \neq w_{j-1}s$ because $w_j < w_j s$. Here the Lemma below is applied to

$$w_{j-1} < w_{j-1}s \neq w_j$$

to obtain $w_{j-1}s < w_j s$.

Thus, altogether, we have a chain

$$v = w_1 s < w_2 s < \ldots < w_{i-1}s = w_i < w_{i+1} < \ldots < w_m = w$$

This gives the corollary. ♣

Lemma: Let $v < w$ with $\ell(v) + 1 = \ell(w)$. If there is $s \in S$ with $v < vs$ (that is , $\ell(v) < \ell(vs)$) and $vs \neq w$, then *both* $w < ws$ and $vs < ws$.

Proof: The proposition above implies that, with our hypotheses, $vs \leq w$ or $vs \leq ws$. The first of these cannot occur, since $\ell(vs) = \ell(w)$ but $vs \neq w$. Since $v \neq w$, $vs \leq ws$ implies $vs < ws$. Then

$$\ell(w) = \ell(v) + 1 = \ell(vs) < \ell(ws)$$

implies that $w < ws$, from the definition of the Bruhat order. ♣

1.9 Special subgroups of Coxeter groups

A **special subgroup** or **parabolic subgroup** of a Coxeter group W with generators S is a subgroup W_T generated by a subset T of S. As is typical here, the notion of special-ness does not make sense without implied or explicit reference to a set of generators of the group.

Since such use of the phrase 'parabolic subgroup' is in conflict with terminology in other parts of mathematics, it is wise to refer to *special* subgroups of Coxeter groups, rather than *parabolic* ones, reserving the latter term for other more important uses.

Proposition: Let (W, S) be a Coxeter system.

- For all subsets T of S, (W_T, T) is a Coxeter system.
- For all subsets T of S,
$$\ell_T = \ell|_{W_T}$$
That is, the length function ℓ_T of W_T with respect to the generators T of W_T is the same as the length function from W with respect to generators S, applied to elements of W_T.
- For any $T \subset S$, if $s_1 \dots s_n$ is a reduced expression for an element of W_T, then all the s_i are in T.
- For any $T \subset S$, a reduced expression for $w \in W_T$ is necessarily already reduced in W.
- For any $T \subset S$, the Bruhat order on W_T is the restriction of the Bruhat order on W.
- The map $W_T \to T$ is an inclusion preserving bijection
$$\{W_T : T \subset S\} \to \{T \subset S\}$$
- For two subsets T and T' of S, we have
$$W_{T \cap T'} = W_T \cap W_{T'}$$
- The set S is a minimal generating set for W.

Proof: Let (W', T) be the Coxeter system with generators T and with Coxeter data
$$m' : T \times T \to \{1, 2, 3, \dots, \infty\}$$
given by the restriction to $T \times T$ of the Coxeter data m for (W, S).

The first assertion is not entirely trivial: while we certainly have a group homomorphism $W' \to W_T$ arising from $T \subset S$ (by the 'universal property' of W', that is , that it has a presentation as a Coxeter group), it is conceivable that this homomorphism could have a proper kernel. We give two proofs, which illustrate different ideas.

The first proof is as follows: suppose some word $t_1 \dots t_n$ in W_T is not reduced (with respect to the generators T of W_T and with respect to *length* in W_T with respect to these generators). Then *a fortiori* it is not reduced in W with respect to the generating set S. Thus, by the Deletion Condition in W,
$$t_1 \dots t_n = t_1 \dots \hat{t_i} \dots \hat{t_j} \dots t_n$$
for some pair of indices i, j. Thus, we see that the group W_T satisfies the Deletion Condition with respect to the generators T. Thus, by Tits' theorem, (W_T, T) is a Coxeter system. And we have already seen that the exponents of products $t_1 t_2$ are indeed what they appear to be. Thus, the Coxeter data for (W_T, T) really is obtained from the data for (W, S), as we desired. This proves that (W_T, T) is a Coxeter system with the expected Coxeter data.

Now we give another proof, the viewpoint of which will also be used in the proof of the other assertions above. Let V' be the real vectorspace with basis

e'_t for $t \in T$, and $V' \to V$ the vectorspace inclusion induced by $T \subset S$, where V has basis e_s for $s \in S$ as before. Let G_T be the subgroup of $GL(V)$ of elements stabilizing V_T, the subspace of V spanned by e_t with $t \in T$. Then we have a commutative diagram

$$\begin{array}{ccc} W' & \to & GL(V') \\ \downarrow & & \uparrow \\ W_T & \to & G_T \end{array}$$

where the vertical arrow on the right is restriction, as is the lower horizontal arrow. The commutativity follows by the *naturality* of all our constructions. Since the top horizontal arrow is an injection (by the previous section), the left vertical arrow must be injective, as well.

Note also that the set-up of the previous paragraph definitively establishes that we may identify V' and V_T as $W' = W_T$-spaces, etc. This is used in the immediate sequel.

To prove that the length functions match, we do induction on $\ell_T(w)$ for $w \in W_T$. If $1 \neq w$, then there is $t \in T$ so that $\ell_T(wt) < \ell_T(w)$. Then, by our comparison of roots and lengths, $we_t < 0$ (in $V' = V_T \subset V$). Then, again invoking the comparison, $\ell(wt) < \ell(w)$. Generally, $\ell(wt) = \ell(w) \pm 1$ and $\ell_T(wt) = \ell_T(w) \pm 1$, so these two inequalities prove that

$$\ell_T(w) = \ell_T(wt) + 1 = \ell(wt) + 1 = \ell(w)$$

invoking the induction hypothesis.

An element $v \in W_T$ has *some* expression as a word in elements of T, so by the Deletion Condition has a *reduced* (in W) expression as a word in elements of T. Thus, by the corollary above (from Bruhat order considerations), *every* reduced (in W) expression for v uses only elements of T, since the set of elements in a reduced expression depends only upon v.

As a variant on the previous assertion and its proof, a reduced expression for $w \in W_T$ is necessarily already reduced in W, since the length functions agree.

Let $\leq'$ be the Bruhat order on W_T. As just noted, any reduced expression for an element $w \in W_T$ involves only elements of T. Then the corollary on subexpressions shows that

$$w \in W \text{ and } w \leq v \iff w \in W_T \text{ and } w \leq' v$$

To prove that

$$W_{T \cap T'} = W_T \cap W_{T'}$$

we need only prove that

$$W_{T \cap T'} \supset W_T \cap W_{T'}$$

since the opposite inclusion is clear. For $w \in W_T \cap W_{T'}$, the set S_v of elements occuring in any reduced expression for w 'can be' a subset of T and 'can be' a subset of T', so, by the corollary on subexpressions, S_v is a subset of $T \cap T'$. Thus, $w \in W_{T \cap T'}$, as desired.

Now let T and T' be distinct subsets of S and show that W_T and W_T' are distinct. By the previous assertion proven, we need only consider the case that $T' \subset T$. Let $s \in T$ but $s \notin T'$. Then (by the subexpression corollary) any reduced expression for s only involves s itself. But then certainly $s \notin W_{T'}$. Thus, $W_{T'}$ is strictly smaller than W_T, as desired. ♣

2. Seven infinite families

- Three spherical families
- Four affine families

Among all possible Coxeter systems (W, S), there are *seven infinite families* of special importance. They fall into two bunches, the first consisting of three families of *spherical* ones, the second consisting of four families of *affine* ones. We will describe these in terms of the Coxeter data. (The terminology *spherical* and *affine* will not be described nor justified until later).

The first bunch consists of three families of *spherical* Coxeter systems, denoted A_n, C_n, D_n. (There is also a B_n, which for our purposes coincides with C_n). The second bunch consists of four families of *affine* Coxeter systems, denoted $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n$.

In the spherical cases the index tells the cardinality of the generating set S, while in the affine cases the index is one less than this cardinality.

A suspicion that there is a connection between A_n and $\tilde{A}_n$, (and likewise with the other letters) is correct, and this relation will be amplified and exploited in the later study of the *spherical building at infinity attached to an affine building*.

2.1 Three spherical families

We will name, give the Coxeter data, and discuss the *occurrence* of three infinite families of Coxeter systems.

The single most popular Coxeter system is the family (or *type*) A_n. This is the system (W, S) with generators $S = \{s_1, \ldots, s_n\}$ where $m(s_i, s_{i+1}) = 3$ and otherwise the generators commute. That is, $s_i s_{i+1}$ is of order 3 while all *other* products $s_i s_j$ with $|i-j| > 1$ are of order 2.

The Coxeter group A_n turns out to be identifiable as the symmetric group permuting $n+1$ things, where s_i is the transposition of the i^{th} and $(i+1)^{\text{th}}$ things. This is not entirely trivial to prove: while it is clear that these transpositions satisfy the relations defining the Coxeter group A_n, it is not so clear that the symmetric group is not a proper quotient of A_n. Anyway, perhaps surprisingly, the identification of A_n with a symmetric group is not of immediate use to us.

The Coxeter system A_n appears later in the study of the *spherical building* attached to $GL(n+1)$. At that point we will find a very indirect proof that the Coxeter group A_n is the permutation group on $n+1$ things.

The Coxeter system of type C_n has generators $s_1, \ldots, s_n$ with data

$$3 = m(s_1, s_2) = m(s_2, s_3) = \ldots = m(s_{n-2}, s_{n-1})$$

while

$$4 = m(s_{n-1}, s_n)$$

and $s_i s_j = s_j s_i$ for $|i - j| > 1$.

The Coxeter group C_n turns out to be identifiable as the *signed permutation group* on n things, although this observation is not so easy to check, and in any case is completely unnecessary for the more serious applications. This group is described as follows: we consider configurations of ordered lists of (e.g.) the numbers 1 through n *and in addition* attach a sign $\pm$ to each. A *signed permutation* is a change in the ordering, together with a change of signs. It is not so hard to check that the *sign change* subgroup, in which no permutations of order but only sign changes occur, is *normal*.

The generators s_i with $1 \leq i < n$ correspond to adjacent transpositions $(i \;\; i+1)$ while s_n corresponds to change-sign of the last item in the ordered list. While it is clear that these items do satisfy the relations defining the Coxeter group C_n, it is not so clear that the signed permutation group is not a proper quotient of C_n.

The Coxeter systems C_n appear in the *spherical building* attached to *symplectic groups* $Sp(n)$ (sometimes denoted $Sp(2n)$), as well as the spherical buildings for other *isometry groups* with the sole exception of certain *orthogonal groups* $O(n, n)$. As in the case of $GL(n+1)$ and A_n, study of these buildings will yield an indirect proof that C_n really is the signed permutation group.

The *oriflamme* Coxeter system D_n has generators which we write as

$$s_1, s_2, s_3, \ldots, s_{n-3}, s_{n-2}, s_n, s_n'$$

with data

$$3 = m(s_1, s_2) = m(s_2, s_3) = \ldots = m(s_{n-3}, s_{n-2})$$

and

$$3 = m(s_{n-2}, s_n) = m(s_{n-2}, s_n')$$

and

$$2 = m(s_n, s_n') \text{ (that is, they } \textit{commute}\text{)}$$

and all other pairs commute. Thus, unlike A_n and C_n, the element s_{n-2} has non-trivial relations with *three* other generators, and concommitantly the Coxeter diagram has a *branch*.

This system occurs in the spherical buildings for orthogonal groups on *even-dimensional* vectorspaces over algebraically closed fields, for example. In this scenario, the construction which turned out nicely for all other isometry groups does not yield a *thick* building, and a slightly different construction is necessary, which engenders this Coxeter system.

In terms of somehow identifying this group in more tangible terms, the best that can be said is that it is identifiable with a subgroup of index two inside a signed permutation group. Luckily, such interpretations are unnecessary.

(The terminology *oriflamme* comes from the drawing of the corresponding Coxeter diagram, as well as schematic drawings of the *flags* involved in the construction of the building, *and* has historical origins in heraldry).

In all these cases, the ambient situation is that in which a group acts on the building so that the 'B' in the corresponding BN-pair is a *parabolic subgroup*.

2.2 Four affine families

We will name, give the Coxeter data, and discuss the *occurrence* of four infinite families of Coxeter systems *in which the group W is infinite*. More specifically, these systems are *affine*, in a sense only clarified later. The spherical case had been appreciated for at least twenty years before the affine phenomenon was discovered.

The simplest affine Coxeter system, which is also the infinite dihedral group, is called $\tilde{A}_1$. It is the system (W, S) with $S = \{s, t\}$ having two generators s, t and st of infinite order (that is, not having finite order). This is the only case among the families we discuss here that some Coxeter datum $m(s, t)$ takes the value $+\infty$. And among *affine* Coxeter groups this is the only group recognizable in more elementary terms.

The description of $\tilde{A}_n$ for $n > 1$ is by generators $s_1, \ldots, s_{n+1}$ where

$$3 = m(s_1, s_2) = m(s_2, s_3) = \ldots = m(s_{n-1}, s_n) = m(s_n, s_{n+1}) = m(s_n, s_1)$$

and all other pairs commute.

Note that the diagram is a *closed polygon* with $n+1$ sides, in light of the last relation (and possibly unexpected) relation $m(s_{n+1}, s_1) = 3$. This feature is anomalous among all spherical or affine systems in the families we care about most. For that matter, this also entails that no one of the generators can be distinguished in any way, apart from the artifact of our ordering. That is, the Coxeter data (or diagram) has a transitive symmetry group itself.

The system $\tilde{A}_{n-1}$ appears in the *affine* building for $SL(n)$ *over a p-adic field*. The corresponding *spherical building at infinity*, as described in the last chapter of this book, is A_{n-1}.

The description of $\tilde{C}_n$ for $n > 1$ is by generators $s_1, \ldots, s_{n+1}$ where

$$m(s_2, s_3) = \ldots = m(s_{n-1}, s_n)$$

and

$$4 = m(s_1, s_2) = m(s_n, s_{n+1})$$

and all other pairs commute. Thus, this differs from the *spherical* (finite) A_{n+1} only in the first and last bits of the Coxeter data, illustrating the sensitivity of the phenomena to the Coxeter data.

Note, also, that the group of symmetries of the data (or of the diagram) is just of order 2, the non-trivial symmetry being reversal of the indexing. This is much smaller than the symmetry group for $\tilde{A}_n$.

The system $\tilde{C}_n$ appears in the *affine* building for $Sp(n)$ and *unitary groups over a p-adic field*. The corresponding *spherical building at infinity*, as described in the last chapter of this book, is C_n.

It is usual to take $\tilde{B}_2 = \tilde{C}_2$. For $n > 2$, the affine $\tilde{B}_n$ is a kind of combination of the *oriflamme* mechanism with the $m(s,t) = 4$ aspect of type C_n, as follows: it has generators which we write as $s_1, s_1', s_3, s_4, \ldots, s_{n+1}$ with relations

$$m(s_1, s_1') = 2 \text{ (that is, commute)}$$
$$3 = m(s_1, s_3) = m(s_1', s_3) = m(s_3, s_4) = \ldots = m(s_{n-1}, s_n)$$

and

$$4 = m(s_n, s_{n+1})$$

Thus, at the low-index end there is a *branching*, while at the high-index end there is a 4 appearing in the data.

This affine *single oriflamme* system occurs in the affine building for orthogonal groups on *odd-dimensional* vectorspaces over p-adic fields, for example.

The last infinite affine family is $\tilde{D}_n$ with $n \geq 4$. This is the *double oriflamme* system, since for example it has the branching at both ends of the data (or diagram). That is, we have generators

$$s_1, s_1', s_3, s_4, \ldots, s_{n-3}, s_{n-2}, s_{n-1}, s_{n+1}, s_{n+1}'$$

with relations

$$3 = m(s_1, s_3) = m(s_1', s_3) = m(s_3, s_4) = \ldots = m(s_{n-2}, s_{n-1})$$
$$3 = m(s_{n-1}, s_{n+1}) = m(s_{n-1}, s_{n+1}')$$

This occurs in the affine building for *certain* orthogonal groups on *even-dimensional* vectorspaces over p-adic fields.

In all these cases, the 'B' in the BN-pair is a *compact open subgroup*, called a *Iwahori subgroup*. This will be explained in detail later when affine buildings and Coxeter systems are defined and examined carefully.

3. Chamber Complexes

- Chamber complexes
- The uniqueness lemma
- Foldings, half-apartments, walls, reflections
- Coxeter complexes
- Characterization by foldings and walls
- Corollaries on foldings and half-apartments

Here our rewriting of group theory as geometry begins in earnest. We make no genuine *direct* use of geometry, but rather develop a vocabulary which is meant to *evoke* geometric intuition. Intuitions suggested must be justified, and this is done below and in the sequel.

Tits' theorem (below) gives a peculiar but important method of constructing Coxeter groups, or of proving that a given group *is* a Coxeter group (with respect to a specified set of generators). In the context of the building theory proper other situations will miraculously deliver the hypotheses of Tits' theorem for *partments in a thick building*.

3.1 Chamber complexes

This section does no more than recall (or set up) standard terminology about *simplicial complexes* and *posets* (partially ordered sets). As noted above, we do not presume any prior knowledge of these things.

In part, in order to *prove* rather than *suggest*, we talk about simplicial complexes as if they were merely some special sort of partially ordered set (*poset*). Of course one is meant to *imagine* that a zero-simplex is a point, a one-simplex is a line, a two-simplex is a triangle, a three simplex is a solid tetrahedron, and so on, and then that these things are stuck together along their faces in a reasonable sort of way to make up a simplicial complex.

Let V be a set, and X a set of *finite* subsets of V, with the property that, if $x \in X$ and if $y \subset x$ then $y \in X$. We also posit that every singleton subset of V lies in X. Then we say that X is a **(combinatorial) simplicial complex** with **vertices** V, and the elements $x \in X$ are **simplices** in X. The *set* of **vertices of a simplex** x in X is nothing other than the *set* x itself.

Remarks: We only consider simplices of *finite dimension*. These are all we will need in subsequent applications, and there are some pointless complications without this assumption.

If $y \subset x \in X$ then y is a **face** of x. In the particular case that $\mathrm{card}(x-y) = 1$ then say that y is a **facet** of x. More generally, if y is a face of x, then the **codimension** of y in x is $\mathrm{card}(x - y)$. The **dimension** of y is $\mathrm{card}(y) - 1$.

Thus, the facets of x are the codimension one faces of x. The relations $y \subset x$ holding in a simplicial complex are the **face relations**.

For a simplex $x \in X$, write $\bar{x}$ for the simplicial complex consisting of the union of x and all faces of x. We may refer to this as the **closure** of x.

Two simplices x, y in a simplicial complex X are **adjacent** if they have a common *facet*.

A simplex x in a simplicial complex X is **maximal** if there is no simplex $z \in X$ of which x is a *proper* face. *In the rest of this book, we will consider only simplicial complexes in which every simplex is contained in a maximal one.* This property follows from an assumption of finite-dimensionality, which we explicitly or implicitly make throughout.

A simplicial complex X is a **chamber complex** if every simplex is contained in a maximal simplex, and if, for all maximal simplices x, y in X, there is a sequence $x_0, x_1, \ldots, x_n$ of maximal simplices so that $x_0 = x$, $x_n = y$, and x_i is adjacent to x_{i+1} for all indices i. If these conditions hold, then maximal simplices are called **chambers**, and the sequence $x_0, \ldots, x_n$ is a **gallery connecting** x **to** y.

A **simplicial subcomplex** of a simplicial complex X is a subset Y of X which is a simplicial complex 'in its own right', that is, with the face relations from X. A **chamber subcomplex** is a simplicial subcomplex which is a chamber complex, and so that the chambers in the subcomplex were maximal simplices in the original complex.

The **distance** $d(x, y)$ from one chamber x to another chamber y is the smallest integer n so that there is a gallery $x = x_0, \ldots, x_n = y$ connecting x to y. A gallery $x_0, \ldots, x_n$ is said to **stutter** if some $x_i = x_{i+1}$.

A chamber complex is **thin** if each facet is a facet of *exactly two* chambers. In other words, given a chamber C and a facet F (codimension one face) of C, there exactly one other chamber C' of which F is also a facet. A chamber complex is **thick** if each facet is a facet of *at least three* chambers.

One fundamental 'example' of simplicial complex is that of a **flag complex** arising from an **incidence geometry**, the latter defined as follows. Let V be a set with a symmetric and reflexive binary relation $\sim$, an **incidence relation**. Then define the **flag complex** X by taking the vertex set to be V itself, and the simplices to be subsets $x \subset V$ so that $h \sim h'$ for all $h, h' \in x$. That is, the simplices are sets of *mutually incident* elements of V. It is easy to check that this procedure really does yield a simplicial complex. Some additional conditions would be necessary to assure that the flag complex arising from an incidence geometry is a chamber complex.

A simplicial complex (with its face relations) gives rise to a partially ordered set (poset) in a canonical manner: the elements of the poset are the simplices, and $x \leq y$ means that x is a face of y. *We will often identify a simplicial complex and its associated poset.*

A **morphism** or **map of simplicial complexes** $f : X \to Y$ is a set map on the set of vertices so that if x is a simplex in X then $f(x)$ (image of the set x of vertices in X) is a simplex in Y. A **retraction** $f : X \to Y$ of X to a subcomplex Y of X is a map of simplicial complexes which, when restricted to Y, is the identity map. If f is a simplicial complex map of X to itself, and if x is a simplex in X, we say that f **fixes** x **pointwise** if $f(v) = v$ for every vertex v of x.

As an example of a morphism of simplicial complexes, if we start with a simplicial complex X, take the canonical poset P associated to X, and then construct the canonical simplicial complex X' associated to P, we will have a (natural) isomorphism $X \to X'$ of simplicial complexes. This is pretty clear.

On the other hand, it is seldom the case that a poset is identifiable as that arising from a simplicial complex. We need further hypotheses. To state the hypotheses succinctly, and for other purposes, we need two definitions. Say that a poset is **simplex-like** if it is isomorphic to the poset of all non-empty subsets of some *non-empty finite* set, with inclusion as the order relation. Say that $z \in P$ is a **lower bound** for $x, y \in P$ if $z \le x$ and $z \le y$. Say that $z \in P$ is a **greatest lower bound** or **infimum** if z is a lower bound for x, y and $z \ge z'$ for every lower bound for x, y. Note that such infimum is unavoidably unique *if it exists*.

Then we have a criterion for a poset to be a simplicial complex:

Proposition: A poset P is obtained as the poset attached to a simplicial complex if and only if two conditions hold: first, that for all $x \in P$ the sub-poset

$$P_{\le x} = \{y \in P : y \le x\}$$

is simplex-like; second, that all pairs x, y of elements of P with *some* lower bound have an *infimum*.

Proof: In one direction this is obvious: thus, we only show that a poset meeting these conditions can be identified with a simplicial complex. Keep in mind that we are supposing throughout that all simplices are *finite* sets of vertices. This is implicit in the definition of *simplex-like*, for example.

First we identify the vertex set. Since all sets $P_{\le x}$ are simplex-like, we may choose a poset isomorphism $f_x : \tilde{S}_x \to P_{\le x}$ where $\tilde{S}_x$ is the poset of all non-empty subsets of a finite non-empty set S_x depending upon x. Thus, $P_{\le x}$ has *minimal elements* x_α: the images of singleton subsets of S_x by f_x. (If there were any doubt, the minimality property is that if $y \le x_\alpha$ then $y = x_\alpha$).

Then x is the supremum, *at least in* $P_{\le x}$, for the set of all the minimal elements less than or equal it, in the sense that if $z \le x$ and $z \ge x_\alpha$ for all these minimal x_α less than or equal x, then $z \ge x$. But it is unclear what happens in the larger poset P.

Let $\xi \in P$ be another element so that $\xi \ge x_\alpha$ for every minimal $x_\alpha \le x$. Since there are elements of P both $\le x$ and $\le \xi$, the two elements x, ξ have an infimum γ. Then $\gamma \ge x_\alpha$ for every one of these minimal x_α. Since $\gamma \le x$, necessarily $\gamma \in P_{\leq x}$, so actually $\gamma = x$ since the structure of $P_{\leq x}$ is so simple. That is, $x \leq \xi$. In other words, *x is the supremum of the set of all minimal elements less than x.*

Note that we *did* use the existence of infima to obtain the uniqueness of the upper bound for the set of minimal elements $x_\alpha \leq x$.

For each minimal element x in P we take a *vertex* v_x, and let the vertex set be

$$V = \{v_x : x \text{ minimal in } P\ \}$$

To each $x \in P$ we associate a set V_x of vertices, consisting of vertices v_y for all minimal $y \leq x$. By the previous discussion, this map is an injection (and the order in P is converted to subset inclusion in the set of subsets of V).

If $\emptyset \neq W \subset V_x$ for some $x \in P$, then since $P_{\leq x}$ is necessarily isomorphic to the set of non-empty subsets of the finite non-empty set V_x, there must be $y \leq x$ whose vertex set V_y is W. ♣

A **chamber complex map** is a simplicial complex map from one *chamber* complex to another which sends chambers to chambers, and which preserves codimensions of faces inside chambers. (If all simplices were *finite-dimensional*, then we could equivalently require that the map preserves *dimensions* of simplices).

A **labeling** or **typing** λ of a poset P is a poset map λ from P to a simplex-like poset L (the **labels** or **types**), so that $x < y$ in P implies $\lambda x < \lambda y$ in L.

We will say that a simplicial complex is **labelable** or **typeable** if the associated poset has a labeling. Note that the condition $x < y \Rightarrow \lambda x < \lambda y$ implies that the label map viewed as a simplicial complex map preserves dimensions. The image under a such label map is the **label** or **type** of the simplex (or of the poset element).

Remarks: Of course, the notion of labeling or typing a simplicial complex is a secondary thing, but is of technical importance. Eventually, when discussing those chamber complexes called buildings, we will show that there is a canonical labeling on thick buildings. Thus, at that point, the notion of labeling can be suppressed further.

If a chamber complex X is labeled by a map $\lambda : X \to L$, we can use a more refined version of *adjacency* of chambers: for $\ell \in L$, say that adjacent chambers C_1, C_2 are **ℓ-adjacent** if $\lambda(C_1 \cap C_2) = \ell$.

One natural way in which a chamber complex X can be typed is if there is a retraction $\lambda : X \to \bar{C}$ of X to $\bar{C}$ for some chamber C in X: the poset of simplices in the simplicial complex $\bar{C}$ is simplex-like. This mechanism comes into play quite often in the sequel.

Remarks: Let S be a set. Let A be the poset of subsets of S with inclusion, and let B be the poset of subsets of S with inclusion *reversed*. Then $A \approx B$ as posets, by the map $x \to S - x$.

Remarks: An *example* of a *chamber system* is given by taking the chambers in a chamber complex, with their adjacency relations, 'forgetting' the rest of the simplicial complex structure. This notion has some utility since, after all, the *cartesian product* of two simplicial complexes is not a simplicial complex (but, rather, is called *polysimplicial*). Addressing the issues in this light is not much more trouble, but is a little more trouble than we need to take.

3.2 The uniqueness lemma

The proof of the following result is what is sometimes called *the standard uniqueness argument*. This little result will be used over and over again, not only throughout our discussion of chamber complexes, but also in discussion of basic facts about buildings, and again later in our finer discussion of the structure of affine buildings and BN-pairs.

Keep in mind that a *facet* is a codimension-one *face*. Note that the hypothesis on the chamber complex Y in the following is somewhat weaker than an assumption that Y is *thin*, although it certainly includes that case. This generality is not frivolous.

Lemma: Let X, Y be chamber complexes, and suppose that every facet in Y is a facet of *at most two* chambers. Fix a chamber C in X. Let $f : X \to Y$, $g : X \to Y$ be chamber complex maps which agree pointwise on C, and both of which send non-stuttering galleries (starting at C) to non-stuttering galleries. Then $f = g$.

Proof: Let γ be a non-stuttering gallery $C = C_0, C_1, \ldots, C_n = D$. By hypothesis, $f\gamma$ and $g\gamma$ do not stutter. That is, $fC_i \neq fC_{i+1}$ for all i, and similarly for g. Suppose, inductively, that f agrees with g pointwise on C_i. Certainly fC_i and fC_{i+1} are adjacent along

$$F = fC_i \cap fC_{i+1} = gC_i \cap gC_{i+1}$$

By the non-stuttering assumption, $fC_{i+1} \neq fC_i$ and $gC_{i+1} \neq gC_i$. Thus, by the hypothesis on Y, it must be that $fC_{i+1} = gC_{i+1}$, since there is no *third* chamber with facet F.

Since there is a gallery from C to any other chamber, this proves that $f = g$ pointwise on all of X. ♣

3.3 Foldings, half-apartments, walls, reflections

The terminology of this section is not quite as standard as the more basic terminology regarding simplicial complexes, but is necessary for the ensuing discussion.

We include several elementary but not entirely trivial lemmas couched in this language. Another version of *reflection* will be discussed at greater length later in preparation for the finer theory of affine buildings.

The last proposition especially will be used over and over again in the sequel.

The theorem of J. Tits proven a little later implies that the results of this section apply to the Coxeter complexes constructed from Coxeter systems (W, S).

The attitude here is that we are trying to play upon our geometric intuition for thin chamber complexes, imagining them to be much like models of spheres or planes put together nicely from triangles.

A **folding** of a *thin chamber complex* X is a chamber complex endomorphism f so that f is a retraction (to its image), and so that f is two-to-one on chambers.

The **opposite folding** g to f (of X), if it exists, is a folding of X so that, whenever C, C' are distinct chambers so that $f(C) = C = f(C')$ then $g(C) = C' = g(C')$. If there is an opposite folding to f, then f is called **reversible**.

Since there is little reason to do otherwise, *here and in the sequel we only concern ourselves with reversible foldings.* Some of these little lemmas do not use such a hypothesis, and some are provable without it, but the whole program is simpler if reversibility is assumed at the outset. Use of reversibility will be noted.

Let f be a folding of a thin chamber complex X. Define the associated **half-apartment** to be the image

$$\Phi = f(X)$$

of a folding. Since f is a chamber complex map, Φ is a sub-chamber-complex of X. For two chambers C, D in X, let $d(C, D)$ be the least integer n so that there is a gallery $C = C_o, \ldots, C_n = D$ connecting C and D. We will use this notation for the following lemmas.

Lemma: There exist adjacent chambers C, D so that $fC = C$ and $fD \neq D$. For any such C, D, we have $fD = C$. Therefore, if γ is a gallery from A to B with $fA = A$ and $fB \neq B$, then $f\gamma$ must *stutter*.

Proof: There are chambers A, B so that $fA = A$ and $fB \neq B$, by definition of a folding. There is a gallery $A = C_o, \ldots, C_n = B$ connecting the two, so there is a least index i so that $fC_i = C_i$ and $fC_{i+1} \neq C_{i+1}$. Take

$C = C_i$ and $D = C_{i+1}$. Let F be the common facet. Since $F \subset C$, $fF = F$. Then fD has $fF = F$ as facet in common with $fC = C$. By the thin-ness of X, this means that fD is either D or C, since those are the only two chambers with facet F. Since $fD \neq D$, we have $fD = C$. ♣

Proposition: The half-apartment Φ is *convex* in the sense that, given C, D both in Φ, *every* minimal gallery $\gamma = C_o, \ldots, C_n$ connecting C, D lies entirely inside Φ.

Proof: Let $\gamma = C_o, \ldots, C_n$ be a minimal gallery connecting C, D. Suppose that some C_i does not lie in Φ. Then there is i so that $C_i \in \Phi$ but $C_{i+1} \notin \Phi$. By the previous lemma, fC_{i+1}. Then $f\gamma$ is a stuttering gallery connecting $C = fC$ and $D = fD$, so can be shortened by eliminating stuttering to give a shorter gallery than γ, contradiction. ♣

Proposition: Let f be a *reversible* folding. Let C, C' be adjacent chambers so that $C \in \Phi$ and $C' \notin \Phi$. Then Φ is the set of chambers D so that $d(C, D) < d(C', D)$.

Proof: Take $D \in \Phi$. Let γ be a minimal gallery from D to C'. Since γ crosses from Φ to its complement, $f\gamma$ stutters, by the above. And $f\gamma$ is a gallery from $D = fD$ to $C = fC'$, so $d(C, D) < d(C', D)$. The other half of the assertion follows by symmetry, using the opposite folding. ♣

Lemma: Let f be reversible. Let C, D be adjacent chambers so that $fC = C = fD$. Let g be another reversible folding of X with $gC = C = gD$. Then $g = f$.

Proof: The previous characterization of the half-spaces fX, gX shows that $fX = gX$. Let $\gamma = C_o, \ldots, C_n$ be a gallery connecting C to D for $D \notin \Phi$. We do induction on n to show that f and g agree pointwise on D. If $n = 1$ then $D = C'$ and the agreement is our hypothesis. Take $n > 1$ and suppose that f and g agree on C_{n-1}, and let x be the vertex of $D = C_n$ not shared with C_{n-1}. Put $F = g(C_{n-1} \cap D) = f(C_{n-1} \cap D)$. Then fC_{n-1} and fD have common facet F; and, $gC_{n-1} = fC_{n-1}$ and gD also have common facet F. By induction, $fC_{n-1} = gC_{n-1}$. By the thin-ness, there are exactly two chambers with facet F. Since f and g are two-to-one on chambers, they must both be injective on chambers not in Φ, so fD cannot be fC_{n-1}; likewise, gD cannot be fC_{n-1}. Therefore, $fD = gD$. ♣

Lemma: Let X be a thin chamber complex. Fix a chamber C_o in X. Let $f : X \to X$ be a chamber complex map so that f fixes C_o pointwise. Let γ be a *non-stuttering* gallery $C_o, C_1, \ldots, C_n$ starting at C. Then either $f\gamma = fC_o, \ldots, fC_n$ stutters, or else f fixes every chamber C_i pointwise.

Proof: Suppose that $f\gamma$ does not stutter. That is, $fC_i \neq fC_{i+1}$ for all i. Suppose, inductively, that f fixes C_i pointwise. Then $C_i = fC_i$ and fC_{i+1}

are adjacent along

$$C_i \cap fC_{i+1} = f(C_i \cap C_{i+1}) = C_i \cap C_{i+1}$$

since the latter intersection is a subset of C_i, which is fixed pointwise by f. Thus, by the thin-ness of X, and by the assumption that $fC_{i+1} \neq C_i$, it must be that fC_{i+1} is the only chamber other than C_i with facet $C_i \cap C_{i+1}$, namely C_{i+1}. ♣

Corollary: There is at most one opposite folding to f.

Proof: If there were an opposite folding f' to f, then the set of chambers in the half-apartment $f'X$ would have to be the complement of the set of chambers in Φ. And, for a pair of adjacent chambers $C \neq C'$ so that $fC = C = fC'$ (shown above to exist), we would have $f'C = C' = f'C'$, by definition. Then the previous lemma gives the uniqueness. ♣

Supposing that f is *reversible*, with opposite f', we define the **associated reflection** $s = s_f = s_{f'}$, as follows. If v is a vertex of X so that $fv = v$, then define $sv = f'v$; if v is a vertex of X so that $f'v = v$, then define $sv = fv$. This defines s as a map on vertices.

Proposition: The reflection s associated to a reversible folding f is an automorphism of X of order 2. For adjacent chambers $C \neq C'$ so that $fC = C = fC'$, this s is the unique non-trivial automorphism fixing the common facet $F = C \cap C'$.

Proof: We need to show that s is a simplicial complex map, that is, that $sx \in X$ for every $x \in X$. Every simplex in X lies in either $\Phi = fX$ or in its complement $f'X$. Since f and f' agree on $fX \cap f'X$, and since f, f' are chamber complex maps, so is s. Since $f \circ f'$ is the identity on fX and $f' \circ f$ is the identity on $f'X$, we have $s^2 = 1$.

If ϕ were another automorphism of X fixing the common facet F pointwise then, by the thin-ness of X, ϕC is either C or C'. In the former case, given a non-stuttering gallery γ starting at C, $\phi\gamma$ certainly does not stutter, since ϕ is injective. Thus, by the uniqueness lemma (3.2)

ϕ is the identity on all chambers in γ. Since this holds for all galleries, ϕ is the identity automorphism of X. If $\phi C = C'$, then the same argument applied to $s \circ \phi$ implies that $s \circ \phi$ is the identity. ♣

A **wall** in X associated to a reflection s (associated to a *reversible* folding) is the simplicial subcomplex in X consisting of simplices fixed pointwise by s. By its definition, a reflection fixes no chamber in X. The above discussion shows that the maximal simplices in a wall are the common facets $C \cap C'$ where C, C' are adjacent chambers interchanged by s.

In this context, a facet lying in a wall is sometimes called a **panel in the wall**.

If C, D are *any* two chambers, and if there is a reversible folding f so that $f(C) = C$ but $f(D) \neq D$, then say that C and D are **separated by a wall** (the wall attached to f and its opposite folding f'). If C, D are *adjacent*, then the common facet $C \cap D$ of C, D is a **panel** (in the wall separating the two chambers). The *reversibility* of the foldings is what makes this a symmetrical relation.

More generally, say that chambers C, D are **on opposite sides of** or are **separated by** a wall (associated to a folding f and its opposite f') if $fC = C$ but $fD \neq D$, or if $fD = D$ but $fC \neq C$. The *reversibility* is what makes this a *symmetric* relationship.

Further, the **two sides** of a wall (associated to a folding f and its opposite f') are the sets of simplices x so that $fx = x$ and $f'x$, respectively.

The **walls crossed by a gallery** $\gamma = C_o, C_1, \ldots, C_n$ are the walls η_i containing the facets $C_i \cap C_{i+1}$, respectively, *under the assumption* that these facets really are panels in walls.

The following explicitly corroborates the intuition suggested by the terminology.

Proposition: Let C, D be chambers in a thin chamber complex. If η is a wall so that C, D are on opposite sides of η, then every minimal gallery from C to D crosses η once and only once. If C, D are on the *same* side of η, then *no* minimal gallery from C to D crosses η.

Proof: The convexity result proven above shows that *some* minimal gallery stays on the same side of η, but we are asking for a little more.

Suppose that C, D are on the same side of a wall η associated to a (reversible) folding f. We may as well suppose (by the reversibility) that $fC = C$ and $fD = D$. If a minimal gallery

$$\gamma = (C = C_o, \ldots, C_m = D)$$

from C to D *did* cross η, then for some index i it must be that C_i and C_{i+1} lie on opposite sides of η. Then $f\gamma$ *stutters*, but is still a gallery from $C = fC$ to $D = fD$. But then we can make a shorter gallery by eliminating the stutter, contradiction.

Suppose that C, D are on opposite sides of η, with associated folding f with $fC = C$ and $fD \neq D$. Let f' be the opposite folding. Then it certainly cannot be that $fC_i = C_i$ for all chambers C_i in a gallery from C to D, not can it be that $f'C_i = C_i$ for all C_i, since $fD \neq D$ and $f'C \neq C$. Thus, *any* gallery from C to D must cross the wall η separating C from D.

Suppose γ crossed η twice. Let i be the smallest index so that $fC_i = C_i = fC_{i+1}$. By the assumption of double crossing, there must also be $j > i$ so that $f'C_j = C_j = f'C_{j+1}$. Take the least such j. Then the gallery

$$(C_o, \ldots, C_{i-1}, C_i, fC_{i+1}, fC_{i+2}, \ldots, fC_j, C_{j+1}, \ldots, C_n)$$

still runs from C to D, but now *stutters* twice, so can be shortened. This shows that a minimal gallery will not cross a wall more than once. ♣

3.4 Coxeter complexes

Let (W, S) be a Coxeter system with S *finite*. In this section we will describe a chamber complex, the **Coxeter complex**, associated to such a pair. At the outset it is not clear that the complex is a simplicial complex at all, much less a chamber complex. That this is so, and other observations, require a little effort. But this effort is repaid now and later by our being able to call upon geometric intuition and heuristics, finally justified by the results of this section.

Incidentally, we also prove that (up to reasonable equivalence), there is a canonical *labeling* of a Coxeter complex. As remarked earlier, this fact allows a certain suppression of this auxiliary notion, if desired.

Let P be the poset of all subsets of W, *with inclusion reversed.* The **Coxeter poset** associated to (W, S) is the sub-poset of P consisting of sets of the form $w\langle T\rangle$ for a *proper* (possibly empty) subset T of S.

The associated **Coxeter complex** $\Sigma = \Sigma(W, S)$ is defined to be the simplicial complex associated to the Coxeter poset of (W, S). That is, $\Sigma(W, S)$ has simplices which are cosets in W of the form $w\langle T\rangle$ for a *proper* (possibly empty) subset T of S, with face relations *opposite* of subset inclusion in W. Of course, when attempting to define a simplicial complex as a poset, there are conditions to be verified to be sure that we really have a simplicial complex. This is done below.

Thus, the maximal simplices are of the form $w\langle\emptyset\rangle = \{w\}$ for $w \in W$, and the next-to-maximal simplices are of the form $w\langle s\rangle = \{w, ws\}$ for $s \in S$ and $w \in W$.

Since $\Sigma(W, S)$ is constructed as a collection of cosets $w\langle S'\rangle$, *there is a natural action of W on* $\Sigma(W, S)$, that is, by left multiplication.

We say that a chamber complex is **uniquely labelable** if, given labelings $\lambda_1 : X \to I_1$ and $\lambda_2 : X \to I_2$ where I_1, I_2 are simplices, there is a set isomorphism $f : I_2 \to I_1$ so that $\lambda_2 = f \circ \lambda_1$, where we also write f for the induced map on subsets of I_2.

Theorem:

- A Coxeter complex $\Sigma(W, S)$ is a *uniquely labelable thin chamber complex.*
- The group W acts by type-preserving automorphisms.
- The group W is transitive on the collection of simplices of a given type.
- The isotropy group in W of the simplex $w\langle S'\rangle$ is $w\langle S'\rangle w^{-1}$.

Proof: Keep in mind that we are not yet justified in calling things 'simplices', because we have not yet proven that we have a simplicial complex: so far, we just have a poset.

It is clear that the maximal simplices are of the form $w\langle\emptyset\rangle = \{w\}$ as noted just above. Since we have seen in discussion of *special subgroups* (1.9), that the map $\langle T\rangle \to T$ is a bijection, the faces of $w\langle\emptyset\rangle$ are all cosets of the form

$w\langle T\rangle$ and are in bijection with proper subsets T of S: if $v\langle T\rangle \supset w\langle\emptyset\rangle$, then $v^{-1}w \in \langle T\rangle$. Thus, we can rewrite the coset $v\langle T\rangle$ as

$$v\langle T\rangle = v(v^{-1}w)\langle T\rangle = w\langle T\rangle$$

as desired.

More generally, given $w'\langle T'\rangle \supset w\langle T\rangle$, it follows that $w'\langle T'\rangle \supset w\langle\emptyset\rangle$, so by the previous paragraph we can rewrite

$$w'\langle T'\rangle = w\langle T'\rangle$$

Thus, $w\langle T'\rangle \supset w\langle T\rangle$, so $\langle T'\rangle \supset \langle T\rangle$, and then $T' \supset T$. That is, the faces of $w\langle T\rangle$ are exactly the cosets $w\langle T'\rangle$ with $T' \supset T$.

Thus, the poset P of cosets $w\langle T\rangle$ in W, with inclusion reversed, is labelable, in the sense of the previous section. Further, given a coset $x = w\langle T\rangle$, we have seen that the collection

$$P_{\leq x} = \{y \in A : y \leq x\}$$

is poset-isomorphic to

$$\{S' \subset S : S' \neq S,\ \ S' \subset T\}$$

That is, this sub-poset is simplex-like, as desired.

Further, given $w_1\langle T_1\rangle, w_2\langle T_2\rangle$ with *some* lower bound $w\langle T\rangle$ in the Coxeter poset, we can find an infimum, as follows. Keep in mind that the ordering in this poset is inclusion *reversed*. We can left-multiply everything by w^{-1}, to assume that the lower bound is of the form $\langle T\rangle$.

Basic facts (1.9) about Coxeter groups and their special subgroups imply that $w_i\langle T_i\rangle \subset \langle T\rangle$ if and only if $T_i \subset T$ and $w_i \in \langle T\rangle$. Thus, we assume these containments for $i = 1, 2$, *and* T is not allowed to be the whole set S.

Let T' be the smallest subset of T so that $T_i \subset T'$ for $i = 1, 2$ and so that $w_2^{-1}w_1 \in \langle T'\rangle$. The existence of such a smallest subset of the finite set T is clear. Then take $w' = w_1$. It is easy to check that this $w'\langle T'\rangle$ contains both sets $w_i\langle T_i\rangle$, so is a lower bound.

On the other hand, from the results mentioned above no *smaller* version of T' will do, since $w_i \in w'\langle T'\rangle$ for $i = 1, 2$ implies that $w_2^{-1}w_1 \in \langle T'\rangle$. And with this choice of T' the condition $w_i\langle T_i\rangle \subset w'\langle T'\rangle$ holds if and only if $(w')^{-1}w_i \in \langle T'\rangle$. This determines w' uniquely up to right multiplication by $\langle T'\rangle$.

Thus, any *other* lower bound $w_o\langle T_o\rangle$ must satisfy $T' \subset T_o$. Thus,

$$w'\langle T'\rangle = w_1\langle T'\rangle \subset w_1\langle T_o\rangle = w_o(w_o^{-1}w_1)\langle T_o\rangle =$$
$$= w_o\langle T'\rangle\langle T_o\rangle = w_o\langle T_p\rangle$$

which establishes that $w'\langle T'\rangle \subset w_o\langle T_o\rangle$.

Thus, the existence of a lower bound implies that there is a *greatest*. (As usual, the uniqueness follows from abstract properties of posets). Thus, by our criterion (3.1) for a poset to be a simplicial complex, the simplicial complex A associated to P is such. *Uniqueness* is proven below.

To prove that A is a (connected) *chamber* complex, we must connect any two maximal simplices by a *gallery*. It suffices to connect an arbitrary maximal simplex $C = w\langle\emptyset\rangle = \{w\}$ to a given one, say $C_o = \{1\}$. Write $w = s_1 \ldots s_n$ with $s_i \in S$. We claim that

$$C_o,\ \ s_1 C_o,\ \ (s_1 s_2)C_o,\ \ (s_1 s_2 s_3)C_o, \ldots,\ \ (s_1 \ldots s_n)C_o$$

is such a gallery. Note the manner in which the s_i appear. Since C_o and $s_i C_o$ are adjacent, their images $(s_1 \ldots s_{i-1})C_o$ and $(s_1 \ldots s_{n-1})s_i C_o$ under left multiplication by $s_1 \ldots s_{i-1}$ are adjacent. Thus, the consecutive chambers in the alleged gallery are adjacent, so it *is* a gallery. Thus, A is a chamber complex.

A next-to-maximal simplex is of the form $\sigma = w\langle s\rangle$. This is a facet of maximal simplices $w'\langle\emptyset\rangle = \{w'\}$ exactly for $w' = w$ and $w' = ws$. That is, each next-to-maximal simplex is a facet of exactly two chambers, so A is *thin*.

Again, our chosen labeling is

$$w\langle T\rangle \to T$$

Then it is clear that the action of W preserves types, and is transitive on the collection of simplices of a given type.

To compute isotropy groups, by the transitivity we may as well consider simplices of the form $\langle S'\rangle$. If $w\langle S'\rangle = \langle S'\rangle$ then $w \in \langle S'\rangle$, and the converse is certainly clear.

Now let us show that the labeling is essentially unique. To this end, we may as well show that *any* labeling λ by subsets of S differs from the labeling $\lambda_o : w\langle T\rangle \to T$ by an automorphism of S. Let π be the permutation of S so that $\pi \circ \lambda = \lambda_o$ on $\bar{C}$, where we identify π with the associated map on subsets of S. We claim that $\pi \circ \lambda = \lambda_o$ on *all* simplices in A.

To see this, it suffices to suppose that π is trivial. We do an induction on the length $\ell(w)$ and consider the simplex $x = w\langle T\rangle$. It suffices to consider the case that x is zero-dimensional. Let $w = s_1 \ldots s_n$ be a reduced expression for w, and let $C_i = s_i \ldots s_n\langle\emptyset\rangle$. Then

$$C = C_o, C_1, \ldots, C_n$$

is a gallery from C to a chamber C_n having x as face. In effect, the induction hypothesis is that λ and λ_o agree on all vertices of $C_o, C_1, \ldots, C_{n-1}$. We may as well consider only the case that x is the unique vertex of C_n *not* shared with C_{n-1}, since otherwise we are already done, by induction.

Let $F = C_{n-1} \cap C_n$. Then $\lambda(x)$ must be a singleton set disjoint from $\lambda(F)$, and $\lambda_o(x)$ must be a singleton set disjoint from $\lambda_o(F)$. Since, by induction, $\lambda_o(F) = \lambda(F)$, it must be that $\lambda_o(x) = \lambda(x)$. This completes the induction step, proving that the labeling is essentially unique.

This establishes all the assertions above. ♣

3.5 Characterization by foldings and walls

The following theorem of Tits gives a fundamental method to 'make' Coxeter groups. While it would be difficult to check the hypotheses of the following theorem without other information, it will be shown later that *apartments* in thick buildings *automatically* satisfy these hypotheses.

The proposition which occurs within the proof is a sharpened variant of the last proposition of the previous section, and is of technical importance in later more refined considerations.

Theorem: A thin chamber complex is a Coxeter complex if and only if any two adjacent chambers are separated by a *wall*.

Remarks: Specifically, we choose a *fundamental chamber* C in the chamber complex X, and the (Coxeter) group W is defined to be the group of simplicial complex automorphisms of X generated by the set S of all reflections through the facets of C. Then (W, S) is a Coxeter system, and the associated Coxeter complex is isomorphic (as chamber complex) to X.

Remarks: The most interesting part of this result is the fact that Coxeter groups can be obtained by constructing thin chamber complexes with some additional properties. At the same time, the assertion that Coxeter complexes have many foldings is a critical technical point which will be used very often later.

Proof: We will show that the pair (W, S) satisfies the Deletion Condition (1.7) , so is a Coxeter system. At the end we will show that, conversely, a Coxeter complex has all the foldings asserted by the theorem.

First, we show that the group W of automorphisms generated by S is transitive on chambers in X. We make the stronger claim that, for all $s_1, \ldots, s_n \in S$, the gallery

$$C, \quad s_1C, \quad s_1s_2C, \ldots, \quad s_1s_2\ldots s_nC$$

is non-stuttering, and that every non-stuttering gallery starting at C is of this form. Indeed, since sC is adjacent to C along $F = C \cap sC$ and w is a chamber map, wsC is adjacent to wC along sF. This proves that this is a gallery. It is non-stuttering since the reflections $s \in S$ fix no chambers.

On the other hand, for D adjacent to wC along the facet $w(C \cap sC)$, $w^{-1}D$ is adjacent to C along $C \cap sC$, so by the thin-ness of X it must be that $w^{-1}D = sC$. Thus, $D = wsC$. By induction on length of the gallery connecting C to D, W is transitive on chambers in X. From an expression $w = s_1, \ldots, s_n$, we get a gallery

$$C, \quad s_1C, \quad s_1s_2C, \ldots, \quad s_1s_2\ldots s_nC$$

from C to wC. Thus, we prove the claim above, and certainly obtain the transitivity of W on the chambers of X.

Next, we construct a retraction $\rho : X \to \bar{C}$, thereby also proving that X is *labelable*, where again $\bar{C}$ is the complex consisting of C and all its faces. Let $C_1, \ldots, C_n$ be the chambers adjacent to C but not equal to C, and let f_i be foldings so that $f_i(C) = C = f_i(C_i)$. Let

$$\psi = f_n \circ f_{n-1} \circ \ldots \circ f_1$$

We *claim* that, given a chamber $D \neq C$, the distance (minimum gallery length) of ψD to C is strictly less than that of D to C. Granting this for the moment, it follows that, for given D, for all sufficiently large n we have $\psi^n(D) = C$. And certainly ψ is the identity on C. Then define

$$\rho = \lim_{n \to \infty} \psi^n$$

Then for any finite set Y of vertices in X there is a finite m so that for all $n \geq m$ we have

$$\rho|_Y = \psi^m|_Y = \psi^n|_Y$$

Thus, this ρ will be the desired retraction.

To prove the claim about the effect of ψ on minimal gallery lengths, it suffices to show that, given a minimal gallery $\gamma = C, C', C'', \ldots, D$ from C to D, $\psi\gamma$ *stutters*, since then there is a shorter gallery obtained by eliminating the stutter. If $f_1\gamma$ stutters, we are done; otherwise, the uniqueness lemma implies that f_1 fixes all chambers in γ pointwise. The same applies to f_2, etc. Thus, if no $f_i\gamma$ stutters, then all the f_i fix γ pointwise. Then $f_iC' = C'$ for all i. But some one of the f_i is the folding that sends C' to C, contradiction. Thus, ψ must cause any gallery from C to $D \neq C$ to stutter, as claimed.

Thus, the retraction $\rho : X \to \bar{C}$ gives a labeling of X by subsets of C. Further, map the poset of subsets of C to the poset of subsets of S by sending the facet F to the reflection s through it. Extend this by

$$F_{i_1} \cap \ldots \cap F_{i_m} \to \{s_{i_1}, \ldots, s_{i_m}\}$$

where s_{i_j} is the reflection through the facet F_{i_j} of C. This is an *inclusion-reversing* isomorphism. Let

$$\lambda : X \to \text{ subsets of } S$$

be the composition of ρ with this map. Then λ is a labeling of simplices in X by subsets of S, but now $x \subset y$ implies $\lambda(x) \supset \lambda(y)$.

Next, we claim that all (reversible) foldings and reflections in X are *type-preserving* (referring to λ). From this it would follow that all elements of W are type-preserving, and that wC and wsC are s-adjacent. Since reflections are pieced together from foldings (that is, from a reversible folding and its opposite) (3.3) , it suffices to prove just that *foldings* preserve type.

Every folding f, by definition, fixes pointwise some chamber C_o. Let D be the closest chamber to C_o so that f might fail to preserve the type of some simplex inside D. Let $C_o, \ldots, C_n = D$ be a minimal gallery connecting C_o to D. By hypothesis, f preserves the type of simplices inside C_{n-1}. In particular, f preserves the type of all the vertices in the common facet $F = C_{n-1} \cap D$. Let x be the vertex of D not contained in F. Since λ and $\lambda \circ f$ are dimension-preserving simplicial complex maps to the 'simplex' (simplex-like poset) of subsets of S with inclusion reversed, neither λx nor $\lambda f(x)$ can lie in $\lambda f(F) = \lambda F$. There is just one vertex not in $\lambda f(F) = \lambda F$, so $\lambda f(x) = \lambda x$. That is, $f(x)$ and x have the same type. By induction, f preserves types. Thus, W preserves types, as claimed.

Next, we show that W acts *simply transitively* on chambers. That is, if $w, w' \in W$ and $wC_o = w'C_o$ for some chamber C_o, then $w = w'$. To prove this, it suffices (as usual) to prove that if $wC = C$ then $w = 1 \in W$. Indeed, if $wc = C$, then since w preserves types it must be that w fixes C *pointwise*. Since w, being an automorphism, can cause no non-stuttering gallery to stutter, it must be that w fixes pointwise any gallery starting at C, by the uniqueness lemma (3.2) . Thus, w fixes X pointwise.

Thus, we see that the map

$$w \to wC$$

is a bijection from W to the chambers of X.

The last proposition of the last section already demonstrated that in a minimal gallery $\gamma = C_o, \ldots, C_n$ from C_o to $C_n \neq C_o$ the walls crossed by γ are distinct, and are exactly the walls separating C_o from C_n. The hypothesis that every facet is a panel in a wall assure that their number is $d(C_o, C_n) = n$.

To see that the Deletion Condition (1.7) holds, a sharper version of the latter observation is necessary.

Since X is typed, we can use the more refined version of adjacency available in a typed simplicial complex, *s-adjacency*. Recall that for $s \in S$ two chambers C_1, C_2 are **s-adjacent** if $\lambda(C_1 \cap C_2) = s$. For example, C and sC are s-adjacent.

We define the **type** of a non-stuttering gallery $\gamma = C_o, \ldots, C_n$ to be the sequence $(s_1, \ldots, s_n)$ where C_{i-1} is s_i-adjacent to C_i. Note that knowledge of the starting chamber of such a gallery and of its type determines it completely.

Proposition: Let $\gamma = C_o \ldots, C_n$ be a non-stuttering gallery of type $(s_1, \ldots, s_n)$. If γ is not minimal (as gallery from C_o to C_n), then there is a gallery γ' from C_o to C_n of type $(s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_n)$.

Proof: The previous observation implies that the number of walls separating C_o from C_n is strictly less than n. Thus, at least one of the walls crossed by γ does not separate the two chambers C_o and C_n, in the sense that they are both in the same half-apartment $\Phi = fX$ of some folding f. But then this wall must be crossed another time, to return to Φ where C_n lies. Thus, repeating a part of the proof of the proposition of the last section, there are indices $i < j$ so that $C_{i-1} \in \Phi$ and $C_j \in \Phi$ but $C_k \notin \Phi$ for all indices k with $i \leq k < j$. Then $fC_i = C_{i-1}$ and $fC_{j-1} = C_j$. Thus, the gallery $f\gamma$ stutters, since $fC_{i-1} = C_{i-1}$ and $fC_j = C_j$. Deleting the repeated chambers gives a strictly shorter gallery from C_o to C_n, as desired. ♣

Finally we can prove that (W, S) has the Deletion Condition. Let $w = s_1 \ldots s_n$ be a non-reduced expression for w. Then

$$\gamma = C, \quad s_1 C, \quad s_1 s_2 C, \quad s_1 s_2 s_3 C, \ldots, \quad s_1 \ldots s_n C$$

is a gallery γ of type $(s_1, \ldots, s_n)$ from C to wC. Since w has a shorter expression in terms of the generators S, there are indices i, j so that there is a shorter gallery γ' from C_o to C_n of type

$$(s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_n)$$

That is, we have concluded that

$$s_1 \ldots s_n C = wC = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_n C$$

Since the map from W to chambers of X by $w' \to w'C$ is a bijection, we conclude that

$$s_1 \ldots s_n = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_n$$

That is, the pair (W, S) satisfies the Deletion Condition, so is a Coxeter system. ♣

What remains is to show that the chamber complex X is isomorphic to the Coxeter complex $\Sigma(W, S)$ attached to (W, S).

It is clear that $\bar{C}$ is a 'fundamental domain' for W on X, that is, any vertex (or simplex) in X can be mapped to a vertex (or simplex) inside $\bar{C}$ by an element of W.

Last, we claim that, for a subset S' of S, the stabilizer in W of the face of C of type S' is the 'parabolic subgroup' $\langle S' \rangle$ of W. Let x be a face of type S'. Certainly all reflections in the facets of type $s \in S'$ stabilize x. Thus, $\langle S' \rangle$ does stabilize x.

On the other hand, we will use induction to prove that, if $wx = x$, then $w \in \langle S' \rangle$. For $w \neq 1$, there is $s \in S$ so that $\ell(w) > \ell(ws)$. Since by now we have a bijection between reduced words and minimal galleries, we obtain a minimal gallery $\gamma = C, sC, \ldots, wC$ from C to wC. From above, the wall η which is the fixed point set of s separates C from wC. Thus, $wx = x$ implies that $swx = sx$. At the same time, $swx \subset swC$ and swC is back in the same half-apartment for s as C. Therefore, $swx = sx$ lies in the fixed-point set η for s; thus, $sx = x$ also, and $swx = x$. By induction on length, $sw \in \langle S' \rangle$. Also, $s \in S'$, since s fixes x pointwise. Then $w = s(sw)$ also lies in $\langle S' \rangle$. This completes the stabilizer computation. ♣

We already know that the chambers of X are in bijection with the chambers in the Coxeter complex, by $wC \to w\langle \emptyset \rangle$, and this bijection is compatible with the action of W (which is simply transitively on these chambers). We attempt to define a chamber complex map by sending a vertex wv of wC, with v a vertex of C of type $S - \{s\}$, to the vertex $w\langle S - \{s\} \rangle$ of $w\langle \emptyset \rangle$. This map and its obviously suggested inverse are *well defined* thanks to the stabilizer computations just above (and earlier for the Coxeter complex). Then this map on vertices extends in the obvious way to a map on all simplices.

This completes the proof that thin chamber complexes wherein any adjacent chambers are separated by a wall are Coxeter complexes. ♣

Now we prove the *converse*, that in a Coxeter complex A any two adjacent chambers C, C' are *separated by a wall*. We must show that, for all C, C', there is a folding f of A so that $f(C) = C$ and $f(C') = C$. We will define this f first just on chambers, and then see that it can thereby be defined on all simplices.

We may suppose that $C = \{1\}$ without loss of generality. Then $C' = \{s\}$ for some $s \in S$. For another chamber $wC = \{w\}$ define $f_o(wC) = wC$ if $\ell(sw) = \ell(w) + 1$, and define $f_o(wC) = swC$ if $\ell(sw) = \ell(w) - 1$. Let H_o be the set of chambers x so that $f_o(x) = x$, and let H'_o be the set of all other chambers.

From the definition, it is clear that $f_o \circ f_o = f_o$. It is merely a paraphrase of the Exchange Condition (1.7) to assert that multiplication by s interchanges H_o and H'_o. The latter fact then implies that f_o is two-to-one on chambers, as required.

A slightly more serious issue is proof that f_o preserves t-adjacency for all $t \in S$. Once this is known we can obtain a simplicial complex map f extending f_o which will be the desired folding. Let wC, wtC be two t-adjacent chambers, and show that f_o sends them to t-adjacent chambers. Either $\ell(wt) = \ell(w) + 1$ or we can reverse roles of w and wt.

In the case that $\ell(sw) = \ell(w) + 1$, we are defining $f_o(wC) = wC$. If $\ell(swt) = \ell(wt) + 1$, then we are defining $f_o(wtC) = wtC$. In this case the t-adjacency is certainly preserved, since nothing moves. If still $\ell(sw) = \ell(w) + 1$ but $\ell(swt) = \ell(wt) - 1$, then $swt = w$. This was proven earlier as an easy corollary of the Exchange Condition (1.7) . Then

$$f_o(wtC) = swtC = wC = f_o(wC)$$

so $f_o(wtC)$ is t-adjacent to $f_o(wC)$ in the degenerate sense that they are *equal*.

In the case that $\ell(sw) = \ell(w) - 1$, the element w admits a reduced expression starting with s, as does wt. Then $f_o(wtC) = swtC$, which is visibly t-adjacent to $f_o(wC) = swC$.

Now we extend the map f_o (which was defined only on *chambers*) to a simplicial complex map, using the preservation of t-adjacency. Fix a chamber wC. Let x be a face of codimension n and of type $\{s_1, \ldots, s_n\}$, where we use the labeling of the Coxeter complex by the set S. By the thin-ness of the Coxeter complex, there is a unique chamber s_i-adjacent to wC, and in fact it is just ws_iC. We claim that

$$f(x) = f_o(ws_1C) \cap f_o(ws_2C) \cap \ldots \cap f_o(ws_nC)$$

Here we invoke the preservation of t-adjacency to be sure that $f_o(ws_iC)$ is still s_i-adjacent to $f_o(wC)$. Thus, the indicated intersection is the unique face of $f_o(wC)$ of type $\{s_1, \ldots, s_n\}$. This is all we need to be sure that this extension preserves face relations, so is a simplicial complex map. ♣

Remarks: We can describe the folding f constructed in the proof more colloquially by saying that it is a retraction to the half-apartment containing the chambers which are closer to C than they are to C', in terms of minimal gallery length. That this is an accurate description follows from the lemmas in the section (3.3) above on foldings and half-apartments.

3.6 Corollaries on foldings and half-apartments

The corollaries below are mere repetitions of lemmas proven earlier in (3.3) regarding foldings and half-apartments, now invoking the theorem of the previous section which assures existence of *foldings and walls in Coxeter complexes.*

Fix adjacent chambers C, C' in a Coxeter complex A, and let $f : A \to A$ be a folding so that

$$f(C) = C = f(C')$$

Existence of f is guaranteed by the previous theorem. Let $H = f(A)$ be the *half-apartment* consisting of all simplices in A fixed by f. We use the somewhat temporary notation $d(x, y)$ for the length of a minimal gallery connecting two chambers x, y in A.

Corollary: Let x, y be two chambers in A, with $f(x) = x$ while $f(y) \neq y$. Let γ be a gallery from x to y. Then $f\gamma$ must *stutter*. ♣

Corollary: The half-apartment H is *convex* in the sense that, given chambers x, y both in H, there is a minimal gallery $\gamma = C_o, \ldots, C_n$ connecting x, y lying inside H, that is, with all $C_i \in H$. ♣

Corollary: The half-apartment H can be characterized as the set of chambers D in A so that $d(C, D) < d(C', D)$. ♣

Corollary: Let g be another folding of X with $g(C) = C = g(C')$. Then $g = f$. ♣

The following two corollaries are of importance in later, more refined, study of the geometry of buildings.

Corollary: Let C, D be chambers in a Coxeter complex. If C, D are on opposite sides of a wall η, then every minimal gallery from C to D crosses η exactly once. Conversely, a gallery from C to D which crosses each wall separating C, D just once, and crosses not others, is minimal. If C, D are on the *same* side of η, then *no* minimal gallery from C to D crosses η.

Proof: The only new thing here (since (3.5) Coxeter complexes have sufficiently many foldings) is the criterion for minimality of a gallery. But since a minimal gallery crosses every separating wall, a gallery which crosses only the separating walls just once and crosses no others has the same length as a minimal gallery. Thus it is minimal. ♣

And we have the variant version of the latter corollary, obtained as a proposition in the course of the proof of the theorem of the last section. A Coxeter complex $\Sigma(W, S)$ is *labelable*, and we may as well suppose that the collection of labels is the generating sets S for W. In particular, let C be the *fundamental chamber* in the Coxeter complex, and for $s \in S$ and $w \in W$ say that chambers wsC and wC are **s-adjacent**

As in the proof of the previous section, we define the **type** of a non-stuttering gallery $\gamma = C_o, \ldots, C_n$ to be the sequence $(s_1, \ldots, s_n)$ where C_{i-1} is s_i-adjacent to C_i. In the previous section we proved the following result for any thin chamber complex wherein any two adjacent chambers are separated by a wall, and we now know that this applies to Coxeter complexes:

Corollary: Let $\gamma = C_o \ldots, C_n$ be a non-stuttering gallery of type $(s_1, \ldots, s_n)$. If γ is not minimal (as gallery from C_o to C_n), then there is a gallery γ' from C_o to C_n of type

$$(s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_n)$$

for some indices $i < j$. ♣

4. Buildings

- Apartments and buildings: definitions
- Canonical retractions to apartments
- Apartments are Coxeter complexes
- Labels, links, maximal apartment system
- Convexity of apartments
- Spherical buildings

The previous work on the group theory and geometry of Coxeter groups was the *local* or *relatively trivial* part of the geometry of buildings, which are made up of Coxeter complexes stuck together in rather complicated ways. But this is not quite the definition we give here, in any case.

The definition we *do* give is misleadingly elementary, and its ramifications are unclear at the outset. The virtue of our definition is that *it can be checked in specific examples*, as we will do repeatedly later.

Thus, our definition does *not* depend upon reference to the material on Coxeter groups or Coxeter complexes, nor even upon the material concerning *foldings* and reflections. Rather, that material is used to *prove* that the present definition does have the implications we want, such as that the apartments are Coxeter complexes.

That is, we give the weakest definition possible, and prove that it still works.

At the end, we can decisively treat the simplest abstract family of examples, called *spherical*, wherein by definition the apartments are *finite* chamber complexes. This is equivalent to the condition that the associated Coxeter groups be *finite*.

4.1 Apartments and buildings: definitions

We use the terminology of (3.3) concerning simplicial complexes, and give the definition of *building* in as simple terms as possible.

A *thick* chamber complex X is called a **(thick) building** if there is a set $\mathcal{A}$ of chamber subcomplexes of X, called **apartments**, so that each $A \in \mathcal{A}$ is a *thin* chamber complex, and

- Given two simplices x, y in X, there is an apartment $A \in \mathcal{A}$ containing both x and y.
- If two apartments $A, A' \in \mathcal{A}$ both contain a simplex x and a chamber C, then there is a chamber-complex isomorphism $\phi : A \to A'$ which fixes both x and C *pointwise*, that is, not only fixes x and C but also fixes all simplices which are faces of x or C.

The set $\mathcal{A}$ is a **system of apartments** in the chamber complex X. Note that we do *not* say *the* apartment system.

Remarks: We will prove below that each apartment in a building is necessarily a *Coxeter complex*. Often (usually?!) this is made part of the *definition* of a building, but this makes the definition unattractive: from a *practical* viewpoint, how would one check that a chamber complex was a Coxeter complex? Yet the fact that the apartments *are* Coxeter complexes is crucial for later developments, so the present definition might be viewed as deceitful, since it does not hint at this. To the contrary, as we will see in our explicit constructions later, our previous preparations indicate that *we need verify only some rather simple properties of a complex in order to prove that it is a building*. In particular, rather than trying to prove that a chamber complex is a Coxeter complex, we will have this fact delivered to us as a consequence of simpler properties.

Remarks: Sometimes half-apartments are called *half-spaces*.

Remarks: We might alter the axioms for a building to *not necessarily* require that the chamber complex X be *thick*, but then we would have to *require* explicitly that there be a system of apartments each of which is a *Coxeter complex*. Then X would be called a **weak building**.

It is convenient to note that a *stronger* (and more memorable, and more symmetrical) version of the second axiom follows immediately:

Lemma: Let X be a thick building with apartment system $\mathcal{A}$. If two apartments $A, A' \in \mathcal{A}$ both contain a a chamber C, then there is a chamber-complex isomorphism $\phi : A \to A'$ which fixes $A \cap A'$ *pointwise*.

Proof: For a simplex $x \in A \cap A'$, there is an isomorphism $\phi_x : A \to A'$ fixing x and C pointwise, by the third axiom. Now our *Uniqueness Lemma* (3.2) implies that there can be at most one such map which fixes C pointwise. Thus, we find that $\phi_x = \phi_y$ for all simplices x, y in the intersection. ♣

Remarks: We can also note that, given two simplices x, y, there is an apartment containing both. Indeed, let C be a chamber with x as a face: that is, C is a maximal simplex containing x. Let D be a chamber containing y. Invoking the axioms, there is an apartment A containing both C and D. Since A itself is a simplicial complex, it also contains x, y.

4.2 Canonical retractions to apartments

For two chambers C, D in a chamber complex Y, let $d_Y(C, D)$ be the *gallery distance* from C to D in Y, that is, the least non-negative integer n so that there is a gallery $C = C_o, \ldots, C_n = D$ from C to D with all C_i in Y. More generally, define the distance $d_Y(x, D)$ from a simplex x to a chamber D as the least non-negative integer n so that there is a gallery $C_o, \ldots, C_n = D$ inside Y with $x \subset C_o$.

Proposition: Let X be a building with apartment system $\mathcal{A}$. Fix an apartment A in $\mathcal{A}$. For each chamber C of A there is a *retraction* $\rho = \rho_{A,C} : X \to A$. Further:

- For a chamber D in A and a face x of C,
$$d_X(x, D) = d_A(x, D)$$
- When restricted to any other apartment B containing C, ρ gives an *isomorphism* $\rho|_B : B \to A$ which is the *identity map* on the overlap $A \cap B$.
- Let C' be another chamber in A, and let B be an apartment containing both C, C'. Then when restricted to B, $\rho_{A,C}$ is equal to $\rho_{A,C'}$.
- This $\rho = \rho_{A,C}$ is the *unique chamber map* $X \to A$ which fixes C pointwise and so that for any face x of C and any chamber D in X
$$d_X(x, D) = d_X(x, \rho D)$$

Remarks: The retraction constructed in the proposition is the **canonical retraction of X to A centered at C.**

Proof: Fix a chamber C in A, and consider another apartment B containing chamber C. Then, by the axioms for a building just above in (4.1) , there is a chamber complex isomorphism $f : B \to A$ fixing C. By the uniqueness lemma (3.2) , for given B there is only one such map.

We claim that, given B, B' with associated f, f', the maps f, f' agree pointwise on the overlap $B \cap B'$. Indeed, let $g : B' \to B$ be the isomorphism which fixes $B' \cap B$ pointwise (by the axioms). Then $f \circ g$ must be f', by the uniqueness observed in the previous paragraph. On the other hand, on $B' \cap B$ the map $f \circ g$ is nothing other than f itself. This proves that the various maps constructed agree on overlaps. This completes the construction of the retraction.

On one hand, clearly
$$d_X(x, D) \leq d_A(x, D)$$
On the other hand, let γ be a minimal gallery from C to D in X. Then apply $\rho : X \to A$ to obtain a gallery of no greater length, lying wholly within A. This proves the assertion about distances from faces of C to chambers within A.

Let x be any face of C, and D another chamber in X. Let γ be a gallery $C_o, \ldots, C_n = D$ with $x \subset C_o$. Let A' be an apartment containing both C and D. Since by construction (above) $\rho|_{A'}$ is an *isomorphism* $A' \to A$, we certainly have $d_A(x, \rho D) = d_{A'}(x, D)$. On the other hand, we just proved that distances within apartments are the same as distances within the building, so
$$d_X(x, \rho D) = d_A(x, \rho D) = d_{A'}(x, D) = d_X(x, D)$$

If $f : X \to A$ were another chamber complex map which fixed C pointwise and preserved gallery lengths, then ρ, f would be maps to a thin chamber complex which agreed pointwise on a chamber and which mapped non-stuttering galleries to non-stuttering galleries. Therefore, by the uniqueness lemma (3.2) , $f = \rho$.

Note that the property that ρ restricted to any other apartment B containing C be an isomorphism follows from the construction. The equality of $\rho_{A,C}$ with $\rho_{A,C'}$ when restricted to an apartment containing both chambers C and C' follows from the construction, together with the uniqueness proven above. ♣

4.3 Apartments are Coxeter complexes

The fact that the apartments in a thick building are unavoidably Coxeter complexes is a corollary of Tits' theorem (3.5) giving a criterion for a thin chamber complex to be a Coxeter complex. This is a primary device for 'construction' of Coxeter groups.

Corollary: The apartments in a (thick) building are Coxeter complexes. Indeed, given an apartment system $\mathcal{A}$ for a thick building, there is a Coxeter system (W, S) so that every apartment $A \in \mathcal{A}$ is isomorphic (as chamber complex) to the Coxeter complex $\Sigma(W, S)$.

Proof: By Tits' theorem (3.5), we need only show that, given two adjacent (distinct) chambers C, C' in an apartment A of the building, there are foldings f, f' so that $fC = C = fC'$ and $f'C = C' = f'C'$. (From our general discussion of foldings in (3.3) and (3.6), this would suffice).

Invoking the *thickness*, let E be another chamber distinct from C, C' with facet $F = C \cap C'$. Let A' be an apartment containing C, E. We use the canonical retractions constructed above in (4.2) , and define $f : A \to A$ to be the restriction to A of $\rho_{A,C'} \circ \rho_{A',C}$. Then, from the definitions of these retractions, $fC = C = fC'$.

We need to prove that f is a folding. Now $\rho_{A,C'}$ preserves distances from any face of C', and $\rho_{A',C}$ preserves distances from any face of C. Since F is the common face of C and C', also f preserves distances from F. In particular, if γ is a minimal gallery $C_o, \ldots, C_n = C'$ with $F \subset C_o$, then $f\gamma$ is non-stuttering.

If $C_o = C$ then $d_X(F, C') = d_X(C, C')$ and, by the uniqueness lemma, f fixes C' *pointwise*. That is, f is the identity map on the subcomplex Y of A consisting of faces of chambers D with $d_X(F, D) = d_X(C, D)$. For D in A, either $C_o = C$ or $C_o = C'$, since A is *thin*. In either case $f\gamma$ starts with C, since $fC' = C$. Then $fD \in Y$. Thus, f is a retraction of A to the subcomplex Y.

Reversing the roles of C and C', we have a retraction f' with $f'C = C' = f'C'$, preserving distances from F, and mapping to the subcomplex Y' consisting of faces of chambers D with $d_X(F, D) = d_X(C', D)$.

Next, we show that Y and Y' have no chamber in common, so that the two partition the chambers of A. Indeed, if D were a common chamber, then both f and f' fix D pointwise. Let γ be a minimal gallery from D to a chamber with face F. Then $f\gamma$ and $f'\gamma$ still are galleries from D to a chamber with face F. Since γ was already minimal, these galleries cannot *stutter*. But then the uniqueness lemma (3.2) shows that $f = f'$. This is certainly not possible: for example, $fC = C \neq C' = f'C$.

It remains to show that f maps the chambers in Y' injectively to Y, and (symmetrically) that f' maps the chambers in Y injectively to Y', since in both cases this proves the two-to-one-ness. The chamber map $f \circ f'$ maps C to itself and fixes F pointwise, so unavoidably fixes C pointwise: the map preserves dimensions, and there is only one vertex of C not inside F. Thus, by the uniqueness lemma (3.2) , $f \circ f'$ is the identity map on Y. Symmetrically, $f' \circ f$ is the identity map on Y'. From this the desired result follows.

Now we prove that all apartments in a given apartment system are isomorphic (as simplicial complexes), from which follows the assertion that they are all isomorphic to a common Coxeter system $\Sigma(W, S)$. Indeed, if two apartments have a common chamber, the building axioms assure that there is an isomorphism from one to the other. (The fact that this isomorphism has additional properties is of no moment right now). Then given two arbitrary apartments A, A', choose chambers C, C' in A, A', respectively. Let B be an apartment containing C, C', as guaranteed by the building axioms. Then B is isomorphic to A and also to A', by the previous remark, so A is isomorphic to A' by transitivity of isomorphic. ♣

4.4 Labels, links, maximal apartment system

In the above there was no discussion of *how anything depended upon the apartment system.* In this section we will see that many things do *not* depend at all upon 'choice' of apartment system, and in fact that *there is a unique maximal apartment system.* This is important for more delicate applications later to spherical and affine buildings. Sometimes this maximal apartment system is called the *complete* apartment system. The notion of *link*, introduced below, is very useful in the proof.

Proposition: A thick building X is labelable in an essentially unique way. That is, given labelings $\lambda_1 : X \to I_1$ and $\lambda_2 : X \to I_2$ where I_1, I_2 are simplex-like posets, there is a set isomorphism $f : I_2 \to I_1$ so that $\lambda_2 = f \circ \lambda_1$, where we also write f for the induced map on subsets of I_2.

Proof: Having seen in (4.3) that the apartments A are Coxeter complexes, we recall from (4.2) that there is a canonical retraction r_C of A to the given chamber C, in effect achieved by repeated foldings of A along the facets of C. This gives *one* labeling of the apartment A by the simplicial complex $\bar{C}$.

And we have already proven (3.4) that the labeling of a Coxeter complex A is essentially unique.

Now we make a labeling of the whole building. Fix a chamber C in an apartment A in an apartment system $\mathcal{A}$ in X. We have the canonical retraction $\rho_{A,C}$ of X to A *centered at* C, as discussed earlier in (4.2) . Then

$$r_C \circ \rho_{A,C}$$

is a retraction of the whole building X to the given chamber C, which gives *one* labeling of the building, extending the labeling of A since $\rho_{A,C}$ is a retraction.

To prove uniqueness, since we know the uniqueness of A, it suffices to prove that there is at most one extension of the labeling $r_C : A \to C$ to a labeling $\lambda : X \to C$. Let D be a chamber in X. Invoking a building axiom (from (4.1)), there exists an apartment A' containing *both* C and D. The essentially unique labeling (3.4) of the Coxeter complex A' implies that the labeling on C (that is, on the simplicial complex $\bar{C}$) completely determines that on A', hence on D (or on $\bar{D}$). Thus, any other labeling is essentially the same as that constructed via the canonical retractions. ♣

Next, we observe that the maps postulated to exist between apartments can be required to preserve labels:

Corollary: For apartments A, A' in a given apartment system with a chamber in common, there is a *label-preserving* chamber-complex isomorphism $f : A \to A'$ fixing $A \cap A'$ pointwise, and *any* isomorphism $f : A \to A'$ fixing $A \cap A'$ pointwise *is* label-preserving.

Proof: The existence of a chamber-complex isomorphism is assured by the building axioms. We need only show that *any* such is unavoidably label-preserving.

Let λ be a labeling of X. Then $\lambda \circ f$ is a labeling on A' which agrees with λ on $A \cap A'$, which by hypothesis contains a chamber. Thus, by the uniqueness of labelings (3.4) of the Coxeter complex A', these labelings must agree. ♣

In a simplicial complex X, the **link** $\mathrm{lk}_X(x)$ of a simplex x is defined to be the subcomplex of X consisting of simplices y so that, on one hand, there is *no* simplex z so that $z \leq x$ and $z \leq y$, but there *is* a simplex w so that $w \geq x$ and $w \geq y$.

Proposition: The link of a simplex in a Coxeter complex $\Sigma(W, S)$ is again a Coxeter complex. In particular, supposing as we may that the simplex x is the face $x = \langle T \rangle$ of the chamber $C = \langle \emptyset \rangle$, then the link of x in $\Sigma(W, S)$ is (naturally isomorphic to) the Coxeter complex of the Coxeter system $(\langle T \rangle, T)$.

Proof: The main point is that there is the obvious poset isomorphism of the link of x with the set

$$\Sigma_{\geq x} = \{ \text{ simplices } z \text{ of } \Sigma(W, S) \text{ so that } z \geq x \ \}$$

by sending y to $y \cup x$ for y a simplex in L. Thus, the link is isomorphic to the poset of special cosets inside W *contained in* $\langle T\rangle$, since the inclusion ordering is reversed. This poset is visibly the poset $\Sigma(\langle T\rangle, T)$, as claimed. ♣

Proposition: The link of a simplex in a thick building is itself a thick building.

Proof: Fix a system $\mathcal{A}$ of apartments in the building X. Let X' be the link of a simplex in X. We propose as apartment system in X' the collection $\mathcal{A}'$ of links of x in apartments in $\mathcal{A}$ containing x. By the previous proposition each link of x in an apartment containing it is a Coxeter complex, so is a thin chamber complex. We must verify the thick building axioms (4.1) .

Given simplices $y, z \in X'$ the simplices $x \cup y, x \cup z$ are contained in an apartment $A \in \mathcal{A}$. Then the link of x in A contains y and z. This verifies one building axiom.

Similarly, for the other axiom, suppose that $B' \in \mathcal{A}'$ were another (alleged) apartment containing both y and z. Let $B \in \mathcal{A}$ be the apartment in X so that B' is the link of x in B. Then B contains both $x \cup y$ and $x \cup z$, so (by the building axiom for X) there is an isomorphism $\phi : B \to A$ fixing $A \cap B$ pointwise. Then the restriction ϕ' of ϕ to B' is an isomorphism $B' \to A'$ fixing $A' \cap B'$ pointwise. This proves the other building axiom.

Regarding *thickness*, let y be a codimension-one face of a chamber in X'. As in the discussion of the link of x in a Coxeter complex, it is immediate that as poset the complex X' is isomorphic to the set $X_{\geq x}$ of simplices in X with face x, by the map $z \to x \cup z$. Thus, the chambers in X' with face y are in bijection with the chambers in the original X with face $x \cup y$. Thus, the thickness of X implies the thickness of the link X'. ♣

Now we use links to prove that the Coxeter system attached to a building is the same for any and all apartment systems.

Theorem: Given a thick building X, there is a Coxeter system (W, S) so that *any* apartment A in *any* apartment system $\mathcal{A}$ is isomorphic to the Coxeter complex $\Sigma(W, S)$.

Proof: We prove that the Coxeter data is determined by the simplicial complex structure of the building. We use a labelling λ of the building by taking λ to be a retraction to a fixed chamber C in a fixed apartment A in X. Let S be the set of reflections in A through the facets of C. Thus, we label a face F of C by the subset of S fixing F.

For distinct $s, t \in S$, let F be a face of type $S - \{s, t\}$, that is, fixed *only* by s and t among elements of S. Then the specific claim is that $m(s, t)$ is the *diameter of the link* $\mathrm{lk}_X(F)$ of F in the building X.

The link $\mathrm{lk}_A(F)$ of F in the apartment A is an apartment in the thick building $\mathrm{lk}_X(F)$. This apartment $\mathrm{lk}_A(F)$ is a Coxeter complex for a Coxeter system whose generating set is just $\{s, t\}$. This is a one-dimensional simplicial complex. It is essentially by *definition* that the diameter of the apartment $\mathrm{lk}_A(F)$ is the Coxeter datum $m(s,t)$.

From above, the link $\mathrm{lk}_X(F)$ is a thick building and that $\mathrm{lk}_A(F)$ is an apartment in it. And the minimal galleries in the apartment are minimal in the whole building, so the diameter of an apartment is the diameter of the whole building.

The latter diameter certainly does not depend upon choice of apartment system. Thus, the Coxeter invariants $m(s,t)$ are determined by the simplicial complex structure of the building, so are the same for *any* apartment system. ♣

Now we can show that there is a unique *maximal* apartment system in *any* thick building.

Corollary: Given a thick building X, there is a *unique* largest system of apartments.

Proof: We make the obvious claim that, if $\{\mathcal{A}_\alpha : \alpha \in I\}$ is a collection of apartment systems $\mathcal{A}_\alpha$, then the union

$$\mathcal{A} = \bigcup_\alpha \mathcal{A}_\alpha$$

is also an apartment system. This would give the proposition. To prove the claim, we verify the axioms (4.1) for apartment systems in a building:

If each apartment $A \in \mathcal{A}_\alpha$ is a thin chamber complex, then certainly the same is true for $\bigcup \mathcal{A}_\alpha$. (We have already seen in (4.3) that each apartment is in fact a Coxeter complex. This, too, is true of the union).

The condition that any two simplices lie in a common apartment is certainly met by the union. The non-trivial axiom to check is the requirement that, given two apartments A, A' with a common chamber C, there is a chamber-complex isomorphism $A \to A'$ fixing every simplex in $A \cap A'$.

Via the lemma, choose a label-preserving isomorphism $f : A' \to A$. Since the Coxeter group W of type-preserving automorphisms of $A \approx \Sigma(W,S)$ is transitive on chambers, we can adjust f so that $f(C) = C$. It is *not* yet clear that this f fixes $A \cap A'$.

On the other hand, let ρ be the retraction of X to A centered at C as in (4.2) , and consider the restriction $g : A' \to A$ of ρ to A'. By definition (3.1) of retraction, g fixes $A \cap A'$. Since A and A' are not necessarily in a common apartment system, we cannot yet conclude that g is an isomorphism of chamber complexes.

But f and g agree on the chamber C, and map to the thin chamber complex A. Let γ be a minimal (necessarily non-stuttering) gallery in A'. The image $f(\gamma)$ is non-stuttering since f is an isomorphism. In our discussion of canonical retractions to apartments (4.2) , we showed that ρ preserves gallery-distances from C, and that γ is minimal not only in the apartment but also in the whole building. Therefore, $g(\gamma)$ also must be non-stuttering. Thus, by the

Uniqueness Lemma (3.2) , we conclude that $f = g$. This verifies the last axiom for a building and an apartment system, proving that the union of apartment systems is an apartment system, thus showing that there is a maximal such. ♣

4.5 Convexity of apartments

The result of this section asserts a *combinatorial convexity* property of apartments.

Proposition: In a thick building X, let A be an apartment containing two chambers C, D. Then any *minimal* gallery in X connecting C, D actually lies inside A.

Proof: Let

$$\gamma = (C = C_o, C_1, \ldots, C_n = D)$$

be a minimal gallery from C to D. If it were not contained in the apartment A, then there would be a chamber C_i in the gallery so that $C_i \in A$ but $C_{i+1} \notin A$. Invoking the *thin-ness* of A, let E be the unique chamber in A distinct from C_i and having facet $C_i \cap C_{i+1}$. Let ρ be the retraction $\rho_{A,E}$ of the whole building to A, centered at E, as defined above in (4.2) . Since this retraction preserves *minimal gallery distances* from E, certainly $\rho(E') \neq E$ for all chambers E' adjacent to E (and not equal to E). In particular, $\rho(C_{i+1}) = C_i$, since the only other possibility is $\rho(C_{i+1}) = E$, which is denied, by the previous remark. Therefore, $\rho(\gamma)$ *stutters*, contradicting the minimality of γ. ♣

4.6 Spherical buildings

A building X whose apartments are *finite* chamber complexes is called a **spherical building**. Likewise, a Coxeter complex which is finite is often called a **spherical complex**.

The thick spherical buildings are the simplest buildings. They are also the most important, appearing *everywhere*. Their theory is relatively elementary, so we can develop much of it immediately. One of the more striking aspects of spherical buildings is the assertion, contained in the last corollary, that there is a *unique apartment system*. This is very special to the spherical case.

The **diameter** of a chamber complex is the supremum of the lengths of minimal galleries $(C_o, \ldots, C_n)$ connecting two chambers. Certainly a *finite* chamber complex has *finite diameter*. (We always assume that chamber complexes (buildings or apartments) are *finite-dimensional*).

Proposition: A thick building of *finite diameter* is *spherical*. A Coxeter complex of finite diameter is finite. The diameter of a building is the diameter of (any one of) its apartments.

Proof: Although we have been supposing always that the generating sets S for Coxeter groups are *finite*, this deserves special emphasis here, since the *dimension* of the Coxeter complex $\Sigma(W, S)$ is one less than the cardinality of S. So finite-dimension of the complex is equivalent to finite generation.

Let C be a chamber in a Coxeter complex $\Sigma(W, S)$ *with S finite*. We already know from (3.4) that, for any $w \in W$, the length of a minimal gallery from C to wC in a Coxeter complex is the length $\ell(w)$ of w. Thus, we are asserting that there is an upper bound N to the length of elements of W. The set S is finite, by the finite dimension of $\Sigma(W, S)$. Let $|S|$ be the cardinality of S. Then there are certainly fewer than

$$1 + |S| + |S|^2 + |S|^3 + \ldots + |S|^N$$

elements in W. Thus, W is finite.

If X is a building with finite diameter N, then any apartment has finite diameter, so is a finite chamber complex, by what we just proved.

Further, if the diameter of X is a finite integer N, then by the axioms there is an apartment A containing two chambers C, D so that there is a minimal gallery *in* X from C to D of length N. Let ρ be the canonical retraction of X to A centered at C. Then the image under ρ of a minimal gallery γ from C to D is certainly not greater than the length of γ. Thus, the diameter of any apartment is *no greater* than the diameter of X.

We have shown that *all apartments are isomorphic* (as chamber complexes). Thus, all their diameters are the same, so must be the same as that of X. ♣

Two chambers in a spherical building are **opposite** or **antipodal** if the length of a minimal gallery from one to the other is the *diameter* of the building.

Proposition: Let C, D be two antipodal chambers in a spherical building X. Let A be any apartment containing both C and D. Then every wall in A separates C, D. And *every* chamber in A occurs in *some* minimal gallery from C to D.

Proof: Of course, the axioms (4.1) for a building assure that there is at least one apartment containing both C, D.

Suppose that C, D lay on the same side of a wall η associated to a folding f and its opposite folding f', as in (3.3) and (3.6) . Without loss of generality we take $fC = C$ and $fD = D$. We claim that $f'D$ is further away from C than D is, in the sense of minimal gallery distances. Indeed, a minimal gallery

$$\gamma = (C + C_o, \ldots, C_n = f'D)$$

from C to $f'D$ must cross η somewhere, in the sense that there is an index i so that $fC_i = C_i = fC_{i+1}$. Then the gallery $f\gamma$ from C to $ff'D = D$ must *stutter*, so is strictly shorter than γ. This contradicts the assumption that C, D were antipodal, thus proving that *all* walls in the apartment A separate the antipodal chambers C, D.

Now let C, D be antipodal, and C' any other chamber in an apartment A containing both C, D. For each wall η in A, the chamber C' must lie on the same side of η as does one or the other of C, D, but not both. We proved earlier in (3.6) that a minimal gallery crosses each separating wall exactly once, and crosses no others. Let

$$\gamma = (C = C_o, \ldots, C_m = C')$$

be a minimal gallery from C to C' and let

$$\delta = (C' = D_o, \ldots, D_n = D)$$

be a minimal gallery from C' to D. Then the set of walls crossed by δ is disjoint from the set of walls crossed by γ, and the union of the two sets is the collection of *all* walls in A.

In particular, the gallery

$$\gamma' = (C = C_o, \ldots, C_m = C' = D_o, \ldots, D_n = D)$$

crosses each wall just once. Thus, by the corollaries (3.6) of Tits' theorem (3.5) on walls and foldings, the gallery γ' is minimal. Thus, the chamber C' appears in a minimal gallery. ♣

As temporary usage, for two chambers C, D in the spherical building X, say that the **convex hull** of this pair is the union of all chambers which lie in some minimal gallery from C to D (and all faces of such chambers).

Corollary: In a thick spherical building X, there is a unique apartment system. In particular, the apartments are the *convex hulls* of pairs of antipodal chambers. Indeed, there is a unique apartment containing a given pair of antipodal chambers.

Proof: Let let C, D be any two antipodal chambers. By the combinatorial convexity of apartments (4.5) , *every* minimal gallery from C to D is contained in *every* apartment containing the two. Thus, the *convex hull* is contained in every apartment containing both C and D. On the other hand, the previous proposition shows that every chamber which lies in *some* apartment containing both C and D occurs in *some* minimal gallery from C to D. Thus, the convex hull of C, D is *the unique* apartment containing the two antipodal chambers. ♣

5. BN-pairs from Buildings

- BN-pairs: definitions
- BN-pairs from buildings
- Parabolic (special) subgroups
- Further Bruhat-Tits decompositions
- Generalized BN-pairs
- The spherical case
- Buildings from BN-pairs

The *original* purpose of construction and analysis of buildings was to provide a systematic geometric technique for the study of *groups* of certain important types.

The notion of BN-pair can be posed without mentioning buildings, and such structures are dimly visible in many examples. Nevertheless, in the end, *verification that given subgroups $B, \mathcal{N}$ of a group G have the BN-pair property is nearly always best proven by finding a building on which G acts nicely.*

The viewpoint taken in this section is that facts about buildings are used to make BN-pairs and prove things about them.

5.1 BN-pairs: definitions

Here we just *define* the notion of **(strict) BN-pair** or **Tits system**. In the next section we will see how BN-pairs arise from group actions on buildings, and later we will *construct* buildings for specific groups. A notion of **generalized BN-pair** will be introduced a little later.

Let G be a group. Suppose that we have subgroups $B, \mathcal{N}$ so that $T = B \cap \mathcal{N}$ is *normal* in $\mathcal{N}$. Let $W = \mathcal{N}/T$, and let S be a set of generators for W.

For $w \in W$, the notation BwB will mean to choose $n \in \mathcal{N}$ so that $nT = w$ in $W = \mathcal{N}/T$, and then put $BwB = BnB$, noting that the latter does not depend on the choice of n, but only upon the coset.

The pair $B, \mathcal{N}$ (more properly, the *quadruple* $(G, B, \mathcal{N}, S)$) is a **BN-pair** in G if

- (W, S) is a Coxeter system.
- Together, $B, \mathcal{N}$ generate G (algebraically).
- **Bruhat-Tits decomposition** $G = \bigsqcup_{w \in W} BwB$ (*disjoint!*)
- $B\langle S'\rangle B = \bigsqcup_{w \in \langle S'\rangle} BwB$ is a subgroup of G, for every subset S' of S, where $\langle S'\rangle$ is the subgroup of W generated by S'.
- $BwB \cdot BsB = BwsB$ if $\ell(ws) > \ell(w)$, for all $s \in S$, $w \in W$
- $BwB \cdot BsB = BwsB \sqcup BwB$ if $\ell(ws) < \ell(w)$
- For all $s \in S$, $sBs^{-1} \not\subset B$. That is, sBs is not contained in B.

The subsets BwB are **Bruhat-Tits cells** or **Bruhat cells** in G. The rules for computing $BwB \cdot BsB$ are the **cell multiplication rules**.

These assertions are stronger than the type of assertion sometimes known as a *Bruhat decomposition*, in subtle but important ways.

5.2 BN-pairs from buildings

This section begins to make one of our main points, applying the elementary results proven so far concerning buildings, to obtain BN-pairs from suitable actions of groups upon buildings. In fact, further and sharper results about the Bruhat-Tits decomposition will follow from the building-theoretic description of it.

Fix a chamber C in an apartment A in an apartment system $\mathcal{A}$ in a (thick) building X, as in (4.1). Assume that X is *finite-dimensional* as a simplicial complex (3.1). We have the canonical retraction $\rho_{A,C}$ of X to A *centered at* C (4.2), and the canonical retraction r_C of A to C (3.4). As noted earlier, the composite

$$\lambda = r_C \circ \rho_{A,C}$$

is a retraction of the whole building X to C labeling (that is, typing) X by C_o, and all other labelings are essentially equivalent to this one (4.4).

Suppose that a group G acts on X *by simplicial-complex automorphisms*, and that G is **type-preserving** in the sense that

$$\lambda \circ g = \lambda$$

for all $g \in G$.

We suppose further that G acts **strongly transitively** on X in the sense that G acts transitively on the set of pairs (A, D) of apartments A and chambers D so that D is a chamber in A.

Remarks: In general, it is necessary to *assume* that the group stabilizes the set of apartments. The following proposition notes that this hypothesis is fulfilled if the apartment system is the *maximal* one. Since in our applications we are exclusively concerned with maximal apartment systems, any more general stabilization question is of little concern to us.

Proposition: If $\mathcal{A}$ is the *unique maximal* system $\mathcal{A}$ of apartments and $f : X \to X$ is a simplicial complex automorphism, then for any $A \in \mathcal{A}$ we have $fA \in \mathcal{A}$.

Proof: The point is, as was shown in discussion of links, labels, and the maximal apartment system (4.4), that there is a *unique* maximal apartment system. It is very easy to check that

$$f\mathcal{A} = \{fB : B \in \mathcal{A}\}$$

is another apartment system in X, so if $\mathcal{A}$ was maximal then unavoidable $f\mathcal{A} = \mathcal{A}$. In particular, $fA \in \mathcal{A}$. ♣

Fix a chamber C_o in a fixed apartment A_o. Let

$$W = \{ \text{ type-preserving automorphisms of } A_o\}$$

$$S = \{ \text{ reflections in codimension-one faces of } C_o\}$$

From Tits' theorem (3.5), (W, S) is a Coxeter system, and A_o is (naturally identifiable with) the associated Coxeter complex.

Define some special subgroups of G:

$$B = \{g \in G : gC_o = C_o\}$$

$$\mathcal{N} = \{g \in G : gA_o = A_o\}$$

$$T = B \cap \mathcal{N}$$

This $(B, \mathcal{N})$ will be the **BN-pair** in G associated to the choice of chamber and apartment (in the chosen system of apartments). (We have yet to prove that it has the requisite properties).

Lemma: The subgroup T acts trivially pointwise on A_o, so is the kernel of the natural map $\mathcal{N} \to W$. Therefore, it is *normal* in $\mathcal{N}$. The induced map

$$\mathcal{N} \to \mathcal{N}/T \subset W$$

is *surjective*.

Proof: From the definitions, it is clear that T *contains* the kernel of the natural map $\mathcal{N} \to W$.

Since T gives maps of the thin chamber complex A_o to itself, trivial on C_o, and not causing any non-stuttering galleries to stutter (since it is injective), by the uniqueness lemma (3.2) it must be that elements of T give the trivial map on A_o. Thus, T maps to $1 \subset W$, so is *equal to* the kernel of $\mathcal{N} \to W$.

On the other hand, given $w \in W$, by the strong transitivity there is $n \in \mathcal{N}$ so that $nC_o = wC_o$. Since n and w are type-preserving, they agree pointwise on C_o, so must give the same effect on A_o, by the uniqueness lemma (3.2). Also, if $n \in B \cap \mathcal{N}$ then n fixes C_o pointwise and so acts trivially on A_o. Therefore,

$$\mathcal{N}/T \approx W$$

as desired. ♣

Remarks: The hypothesis of strong transitivity assures that varying the choice of $C_o \subset A_o$ merely conjugates the BN-pair. In particular, in group-theoretic terms, this means that any other choice of apartment changes $\mathcal{N}$ just by conjugation by some element of B.

Corollary: All possible groups $T = \mathcal{N} \cap B$ inside a fixed B, for varying choices of A_o and $\mathcal{N}$, are conjugate to each other by elements of B (not merely by elements of G). ♣

Keep notation as above, with fixed pair $C_o \subset A_o$. For $S' \subset S$, let $F_{S'}$ be the face of C_o whose stabilizer in W is $\langle S' \rangle$. Let

$$P_{S'} = \text{ stabilizer of } F_{S'} \text{ in } G$$

This is the **standard parabolic subgroup of G of type S'**. Note that with $S' = S$ we obtain the whole group G as (improper) parabolic subgroup

$$G = P_S$$

(in a degenerate sense, since $W = \langle S \rangle$ stabilizes only the empty set) and with $S' = \emptyset$ obtain the **minimal standard parabolic subgroup**

$$B = P_\emptyset$$

Remarks: Yes, there is conflict between the present use of *parabolic* subgroup and the use of the same phrase for *special* subgroups of Coxeter groups (1.9). This is why use of *special subgroup* in the Coxeter groups situation is preferable.

Theorem: The quadruple $(G, B, \mathcal{N}, S)$ satisfies the axioms for a BN-pair. Beyond what we have already noted, this explicitly includes

- **Bruhat-Tits decomposition** Each standard parabolic subgroup $P_{S'}$ of G, including $G = P_W$ itself, has a decomposition
$$P_{S'} = \bigsqcup_{w \in \langle S' \rangle} BwB$$
- $BwB \cdot BsB = BwsB$ if $\ell(ws) > \ell(w)$, for all $s \in S$, $w \in W$.
- $BwB \cdot BsB = BwsB \sqcup BwB$ if $\ell(ws) < \ell(w)$.
- For all $s \in S$, $sBs^{-1} \not\subset B$, that is, sBs is not a subset of B.
- And for $g \in G$ the coset BwB is determined by
$$\rho_{A_o, C_o}(gC_o) = wC_o$$
where ρ_{A_o, C_o} is the canonical retraction of X to the apartment A_o centered at C_o.

Remarks: Only the last assertion, which gives a finer explanation of the Bruhat-Tits decomposition, uses an explicit reference to the building and the action of the group upon it. So if such information is not needed it is possible to describe the group-theoretic consequences of the building-theory without any mention of the buildings themselves.

Remarks: Of course, similar properties hold for $BsB \cdot BwB$ as asserted above for $BwB \cdot BsB$. Implicit in the above is that the unions

$$\bigsqcup_{w \in \langle S' \rangle} BwB$$

are indeed subgroups of G. Also implicit is the assertion that

$$(\mathcal{N} \cap P_{S'})/T = \langle S' \rangle \subset W$$

Proof: First we prove the Bruhat decomposition for the standard parabolic subgroups. Given $g \in P_{S'}$, choose an apartment A containing both C_o and gC_o, and by strong transitivity take $b \in B$ so that $bA = A_o$. Then $bgC_o = wC_o$

for some $w \in W$, by the transitivity of W on the chambers in the apartment A_o. So $bg \in wB$, and $g \in BwB$. Further, since $g \in P_{S'}$ and $B \subset P_{S'}$, this w is in $F_{S'}$. This proves that

$$P_{S'} = \bigcup_{w \in \langle S' \rangle} BwB$$

To prove disjointness of the unions above, we need only prove

$$G = \bigsqcup_{w} BwB$$

Multiplication by the element b (in the notation above) gives an isomorphism $A \to A_o$ fixing C_o pointwise. By the uniqueness lemma, there is only one such, the retraction $\rho = \rho_{A_o, C_o}$ to A_o centered at C_o considered earlier (4.2).

The discussion just above shows that $g \in BwB$ where w is the uniquely determined element $w = f(g)$ of W so that $\rho(gC_o) = wC_o$, proving the very last assertion of the theorem. (Recall the *simple* transitivity of W on the apartments). We need to show that $f(BwB) = w$. Take $n \in \mathcal{N}$ so that $nT = w$. For $b, b' \in B$, letting $g = bnb'$,

$$gC_o = bnb'C_o = bnC_o = bwC_o \in bA_o$$

Left multiplication by b^{-1} gives an isomorphism of bA_o to A_o fixing C_o pointwise, so it must be (by uniqueness of ρ) that

$$\rho(gC_o) = b^{-1}(gC_o) = wb'C_o = wC_o$$

Thus, $f(bnb') = w$. This proves the *disjointness* in the Bruhat-Tits decomposition.

Next, for $s \in S$ and $w \in W$, we consider products

$$BwB \cdot BsB = \{b_1 w b_2 s b_3 : b_1, b_2, b_3 \in B\}$$

In *any* group G it would be true that such a product would be a union of double cosets BgB, since it is stable under left and right multiplication by B. Further, certainly $ws \in BwB \cdot BsB$, so this product of double cosets always contains $BwsB$.

Now we prove, first, that

$$BwB \cdot BsB \subset BwsB \cup BwB$$

Recall that the retraction $\rho : X \to A_o$ (as just above) is type-preserving, so also preserves s-adjacency of chambers in the sense of (3.1). So the function $f : G \to W$ defined just above (in terms of ρ) satisfies

$$f(gh) = f(g) \text{or } f(g)s$$

for all $g \in G$ and $p \in P_{\langle s \rangle}$, where (again)

$$P_{\langle s \rangle} = B \sqcup BsB$$

is the stabilizer in G of the face F of C_o fixed by s. Thus,

$$f(BwB \cdot BsB) \subset f(BwB) \sqcup f(BwB)s = w \sqcup ws$$

so that

$$BwB \cdot BsB \subset BwB \sqcup BwsB$$

as asserted.

Suppose that $\ell(ws) > \ell(w)$. We claim that in this case $BwB \cdot BsB = BwsB$. It suffices to show that in this case $wBs \subset BwsB$. Take $n, \sigma \in \mathcal{N}$ so that $nT = w$ and $\sigma T = s$. Given $g = nb\sigma \in nB\sigma$, we must show that $\rho(gC_o) = wsC_o$, with the retraction ρ as above.

Now

$$gC_o = nb\sigma C_o = nbsC_o$$

is s-adjacent to $nbC_o = nC_o = wC_o$ and is *distinct* from it. Let

$$\gamma_o = C_o, C_1, \ldots, wC_o = nC_o$$

be a minimal gallery from C_o to $nC_o = wC_o$, and let

$$\gamma = C_o, C_1, \ldots, wC_o, nbsC_o$$

We grant for the moment that γ is a *minimal* gallery. Since $\rho(nb\sigma C)$ is s-adjacent to $\rho(nbC) = \rho(wC) = wC$, $\rho(nb\sigma C)$ is either wC or wsC, since these are the only two chambers in A_o with facet F. If $\rho(nb\sigma C) = wC$ then $\rho(\gamma)$ would *stutter*, contradicting the fact that ρ preserves distances (4.2), using the minimality of γ. Thus, $\rho(nb\sigma C) = wsC$.

It remains to show that γ is minimal, assuming $\ell(ws) > \ell(w)$. Let ρ' be the retraction to A_o centered at wC_o. Since ρ' preserves distances from wC and $nb\sigma C_o \neq C_o$, it must be that $\rho'(nb\sigma C_o) \neq wC_o$. Thus, since ρ' also preserves s-adjacency (being type-preserving), $\rho'(nb\sigma C_o) = wsC_o$. Thus,

$$\rho'(\gamma) = \rho'(C_o), \ldots, \rho'(wC_o), \rho'(nb\sigma C_o) =$$

$$= C_o, \ldots, wC_o, wsC_o$$

The part $\rho'(\gamma_o)$ of $\rho'(\gamma)$ going from C_o to wC_o is minimal, since ρ' preserves distances from wC_o and γ_o was assumed minimal. Thus, since $\ell(ws) = \ell(w) + 1$, the gallery $\rho'(\gamma) = C_o, \ldots, wC_o, wsC_o$ in A_o is minimal, where we use the correspondence between word-length and gallery-length holding in any Coxeter complex (3.4). Thus, necessarily γ is minimal, since its image by ρ' is minimal.

Next we show that $s^{-1}Bs \not\subset B$. Since X is thick, for every $s \in S$ there is another chamber C' distinct from C_o and sC_o which is s-adjacent to C_o. Let F be the facet $C_o \cap sC_o$ of type s. There is $g \in G$ so that $gC_o = C'$, since G is transitive on chambers. Since g is type-preserving g must fix F. That is,

$$g \in P_{\langle s \rangle} = B \sqcup BsB$$

Since $gC_o \neq C_o$, $g \notin B$, so $g \in BsB$. Also, $gC_o \neq sC_o$, so $g \notin sB$. Thus, we have shown that $BsB \not\subset sB$, so that necessarily $Bs \not\subset sB$, or $s^{-1}Bs \not\subset B$.

Last, we consider the case $\ell(ws) = \ell(w) - 1$ and prove the other cell multiplication rule

$$BwB \cdot BsB = BwB \sqcup BwsB$$

What remains to be shown in order to prove this is that $w \in BwB \cdot BsB$. By the previous paragraph, we already know that $sBs \not\subset B$ for $s \in S$, so

$$B \neq BsB \cdot BsB$$

But we have shown that

$$B \sqcup BsB \supset BsB \cdot BsB$$

Thus, evidently

$$(BsB \cdot BsB) \cap BsB \neq \emptyset$$

so must be all of BsB since the intersection is left and right B-stable. In particular,

$$s \in BsB \cdot BsB$$

Assume $\ell(ws) = \ell(w) - 1$. This is the same as

$$\ell(ws \cdot s) = \ell(ws) + 1$$

so we can apply the earlier result in this direction, to obtain

$$BwsB \cdot BsB = BwssB = BwB$$

Multiplying by BsB gives

$$BwsB \cdot BsB \cdot BsB = BwB \cdot BsB$$

The left-hand side contains

$$ws \cdot BsB \cdot BsB = ws(B \sqcup BsB)$$

which contains $ws \cdot s = w$. Thus, for $\ell(ws) = \ell(w) - 1$,

$$BwB \cdot BsB = BwB \sqcup BwsB$$

as claimed. ♣

5.3 Parabolic (special) subgroups

In this section we do *not* use any hypothesis that the BN-pair arises from a strongly transitive action on a thick building.

The phenomena surrounding the *parabolic* or *special* subgroups described here constitute a unifying abstraction which includes *literal* parabolic subgroups, as well as certain compact open subgroups called *Iwahori* and *parahoric* subgroups. These specific instances of the general idea play a central role in applications. (See chapter 17).

Let G be a group possessing a triple $B, \mathcal{N}, S$ as above (forming a BN-pair). Again, a subgroup P of G is a **(standard) 'parabolic'** or **(standard) 'special'** subgroup (with respect to $B, \mathcal{N}$) if it is one of the subgroups

$$P_{S'} = \bigsqcup_{w \in \langle S' \rangle} BwB$$

Since study of Coxeter groups shows (1.9) that $S' \to \langle S' \rangle$ is an order-preserving injective map, from the defining properties of a BN-pair we see that $S' \to P_{S'}$ *is an injective map.*

More generally, a subgroup of G is called a **parabolic subgroup** if it is *conjugate* in G to one of the standard parabolic subgroups (with respect to $B, \mathcal{N}, S$).

Proposition: Let $w = s_1 \ldots s_n$ be a reduced expression. Then the smallest subgroup of G containing BwB contains s_i for all i. It is also generated by B and $w^{-1}Bw$.

Proof: From the cell multiplication rules (5.1),

$$Bs_1B \cdot Bs_2B \cdot \ldots \cdot Bs_nB = BwB$$

Thus, the subgroup P of G generated by B and w is contained in the subgroup generated by B and all the s_i. We will prove by induction on $n = \ell(w)$ that each s_i is in P, which will prove both assertions of the proposition.

Since $\ell(s_1 w) < \ell(w)$, from the cell multiplication rules we know that $s_1 Bw$ meets BwB, so $s_1 B$ meets $BwBw^{-1}$, and

$$s_1 \in BwBw^{-1}B$$

Therefore, P certainly contains $s_1 wBw^{-1}s_1$. Applying the induction hypothesis to the shorter element $s_1 w$ gives the result. ♣

Corollary: The parabolic subgroups of G are exactly those subgroups containing B. Every parabolic subgroup is its own normalizer in G, and no two are conjugate in G. For a subgroup P of G containing B, let $W_P = (P \cap \mathcal{N})T/T$. Then we have

$$P = BW_PB$$

Proof: If a subgroup P of G contains B, then it is a union of double cosets BgB. Invoking the Bruhat-Tits decomposition, we may as well only consider double cosets of the form BwB with $w \in W$ (or, more properly, in $\mathcal{N}$). Let

$$W' = \{w \in W : BwB \subset P\}$$

Then certainly $P = BW'B$. Since $Bww'B \subset BwB \cdot Bw'B$ and

$$Bw^{-1}B = \{g^{-1} : g \in BwB\} = (BwB)^{-1}$$

we see that W' is a subgroup of W. The proposition assures that W' contains all the elements of S occurring in any reduced expression for any of its elements, so W' is the 'special' or 'parabolic' subgroup of W (now in the Coxeter group sense (1.9) of these words) generated by $S' = S \cap W'$. Therefore, P is a parabolic subgroup of G (in the present sense of the word).

Suppose that $gPg^{-1} = Q$ for two parabolic subgroups P, Q. Let $w \in W$ so that $g \in BwB$. Then $wPw^{-1} = Q$, so

$$wBw^{-1} \subset wPw^{-1} \subset Q$$

Therefore, as $B \subset Q$, from the proposition we see that $BwB \subset Q$. Thus, $g \in Q$, and then $P = Q$. ♣

Remarks: This corollary shows that the notion of *special* or *parabolic* subgroup does not depend upon the choice of S. Indeed, in light of the corollary, we can now correctly refer to these subgroups $P = BW_PB$ as *parabolic subgroups containing* B.

5.4 Further Bruhat-Tits decompositions

Now we *do* assume that our BN-pair in the group G is obtained from a strongly transitive action on a thick building X, in order to give geometric arguments rather than more purely combinatorial. We assume that X is *finite-dimensional*, so that the set S of generators for the Coxeter system is *finite*. Keep the notation above. Let $P_1 = BW_1B$ and $P_2 = BW_2B$ be parabolic subgroups (containing B), where $W_i = \langle S_i \rangle$ for two subsets S_1, S_2 of S.

Theorem: We have a bijection

$$W_1 \backslash W / W_2 \leftrightarrow P_1 \backslash G / P_2$$

given by $W_1 w W_2 \leftrightarrow P_1 w P_2$.

Proof: Let $\mathcal{N}$ be the subgroup of G which, modulo $T = B \cap \mathcal{N}$, is W. As usual, we need not distinguish between $\mathcal{N}$ and W when discussing B-cosets.

Starting from the Bruhat-Tits decomposition $G = \bigsqcup_w BwB$, given $g \in G$ we can left multiply by some element b_1 of $B \subset P_1$ and right multiply by some element b_2 of $B \subset P_2$ so that $b_1 g b_2 \in W$. Then we surely may further multiply on the left by W_1 and on the right by W_2.

On the other hand, we need to show that $w' \in P_1 w P_2$ implies that $w' \in W_1 w W_2$. Let F_i be the face of C_o of type S_i, that is, with stabilizer $P_{S_i} = P_i$.

Given $g \in G$, let A be an apartment containing both F_1 and gF_2, by the axioms (4.1). We claim that there is an element $p \in P_1$ so that $pA = A_o$. Indeed, let C be a chamber of A with face F_1. There is $h \in G$ so that $hC = C_o$, by transitivity of G on chambers in X. Since both C and C_o have just the one face (that is, F_1) of type S_1, necessarily $hF_1 = F_1$. That is, $h \in P_1$. Then hA and A_o both contain C_o, so by *strong* transitivity there is $b \in B$ so that $bpA = A_o$. Then $bp \in P_1$ is the desired element, proving the claim.

Further, the conditions $pF_1 = F_1$ and $pA = A_o$ determine p uniquely left modulo

$$H = \{q \in G : qA_o = A_o \ \text{ and } \ qF_1 = F_1\}$$

Certainly $T \subset H$, and we have

$$H/T = \langle S_1 \rangle = W_1$$

Then $pgF_2 = wF_2$ for some $w \in W$, since W acts transitively on simplices in A_o of a fixed type. Let $n \in \mathcal{N}$ be such that $nT = w$. Note that, given g

and p, w is uniquely determined right modulo $W_2 = \langle S_2 \rangle$. Then we have

$$g \in P_1 n P_2 = P_1 w P_2$$

The ambiguity in choices of p and w is that we may replace p, n by $n_1 p, n_1 n n_2$ for $n_1 \in H$ and $n_2 \in W_2$.

Therefore, if $P_1 w P_2 = P_1 w' P_2$, then both $w' \in P_1 w P_2$ and $w' \in P_1 w' P_2$. The qualified uniqueness just proven shows that $W_1 w W_2 = W_1 w' W_2$, as desired. ♣

5.5 Generalized BN-pairs

In use, it is important to be able to drop the condition that the group acting *preserve types* or *labels* in its action upon the building X. This entails some complications in the previous results, which we now explain. Throughout, the idea is to reduce the issues to the case of a *strict* BN-pair, that is, a BN-pair in the sense discussed up until this point. Emphatically, we are assuming that the set S is *finite*, which is equivalent to the assumption that the building X is *finite-dimensional* as a simplicial complex.

Let X be a thick building, and let a group $\tilde{G}$ act upon it by simplicial complex automorphisms. Further assume that $\tilde{G}$ stabilizes the set of apartments.

Remarks: As earlier, we need to explicitly *assume* that the action of $\tilde{G}$ stabilizes the set of all apartments. Later we will show that this is often *automatic*, and in any case is *visibly true* in most concrete examples.

Fix a chamber C_o and an apartment A_o containing it. Let $\lambda : X \to C_o$ be a retraction of the building to C_o, as earlier, giving a type-ing (labeling) of X. Let G be the subgroup of $\tilde{G}$ *preserving types*, that is,

$$G = \{g \in \tilde{G} : \lambda \circ g = \lambda\}$$

We assume that the subgroup G of $\tilde{G}$ is itself strongly transitive.

As usual, let B be the stabilizer in G of C_o, let $\mathcal{N}$ be the stabilizer in G of A_o, and $T = B \cap \mathcal{N}$. Thus, we have a *strict* BN-pair in G.

Also, let $\tilde{B}$ be the stabilizer in $\tilde{G}$ of C_o, let $\tilde{\mathcal{N}}$ be the stabilizer in $\tilde{G}$ of A_o, and $\tilde{T} = \tilde{B} \cap \tilde{\mathcal{N}}$.

From our results on thick buildings (4.3), the apartment A_o is the Coxeter complex associated to (W, S), where $W = \mathcal{N}/T$ and where S consists of reflections through the facets of the chamber C_o. (Recall that, in the course of other proofs, we have seen that T is a normal subgroup of $\mathcal{N}$ and acts pointwise trivially on all of A_o. The latter follows from the type-preserving property and by invoking the uniqueness lemma (3.2)).

Keep in mind that the *strict* BN-pair properties (5.1) entail Bruhat-Tits decompositions

$$G = \bigsqcup_{w \in W} BwB$$

We proved in (5.2) that this situation *does* arise from a group action as we have presently. And, more generally (5.4),

$$B\langle S'\rangle B = \bigsqcup_{w\in\langle S'\rangle} BwB$$

is a *subgroup* of G, for every subgroup S' of S, where $\langle S'\rangle$ is the subgroup of W generated by S'. Conversely, every subgroup of G containing B is of this form, with uniquely determined S', and is its own normalizer (5.3). For $s \in S$ and $w \in W$, we have cell multiplication rules (5.1)

$$BwB\, BsB = BwsB \quad \text{for} \quad \ell(ws) > \ell(w)$$

$$BwB\, BsB = BwsB \sqcup BwB \quad \text{for} \quad \ell(ws) < \ell(w)$$

For all $s \in S$, $sBs^{-1} \not\subset B$.

The following theorem contains some non-trivial assertions about $\tilde{G}$ in relation to the strict BN-pair $(G, \mathcal{N}, B)$. These assertions, together with the strict BN-pair results on $(G, \mathcal{N}, B)$, tell almost everything we need about the 'generalized' BN-pair $(\tilde{G}, \tilde{\mathcal{N}}, B)$.

Remarks: Note that although $\tilde{B}$ is defined here, its type-preserving subgroup B is the item of consequence.

Theorem:

- The groups $\mathcal{N}, B$ are normalized by $\tilde{T}$, and conjugation by elements of $\tilde{T}$ stabilizes S, as automorphisms of A_o. We have $\tilde{\mathcal{N}} = \tilde{T}\mathcal{N}$ and $\tilde{B} = \tilde{T}B$.
- The group G is a normal subgroup of $\tilde{G}$, of finite index, and $\tilde{G} = \tilde{T}G$.
- With $\Omega = \tilde{T}/T$, $\tilde{\mathcal{N}}/T$ is a semi-direct product $\Omega \times W$ with normal subgroup W. Also, $\tilde{G}/G \approx \Omega$.
- For $\sigma \in \Omega$ and $w \in W$, we have $\sigma w \sigma^{-1} \in W$. And $\sigma B = B\sigma = B\sigma B$ and
$$\sigma BwB = B\sigma wB = B(\sigma w \sigma^{-1})B\sigma$$

Proof:

Lemma: If $g \in \tilde{G}$ has the property that it preserves types of the faces of a chamber C_1, then $g \in G$.

Proof: Let A be any apartment containing the chamber C_1 on which g preserves types, and let $A_2 = gA$ and $C_2 = gC_1$. Take $h \in G$ so that $hC_2 = C_1$ and $hA_2 = A$, invoking the strong transitivity of G. Then the type-preserving property of g just on C_1 implies that hg is the identity on C_1 pointwise (that is, on all faces of C_1, that is, on all vertices of C_1). Then hg is a map from the *thin* chamber complex A to itself which, being an automorphism of X, does not cause any non-stuttering gallery to stutter. Thus, invoking our uniqueness lemma (3.2), since hg is trivial on C_1, it must be that hg is trivial on all of A.

That is, hg certainly preserves types on A. Thus, $g = h^{-1}(hg)$ as a map $A \to A_2$ preserves types on A. Now A was an arbitrary apartment containing

C_1, and any chamber lies in an apartment also containing C_1 (by the building axioms (4.1)), so g preserves types on all of X. This is the lemma. ♣

Next, we prove that the group $\tilde{T}$ normalizes B. Let $t \in \tilde{T}$. For $b \in B$ and for a vertex v of C_o,

$$t^{-1}bt(v) = t^{-1}(b(tv)) = t^{-1}t(v) = v$$

since B acts *pointwise* trivially on C_o. That is, $t^{-1}bt$ acts pointwise trivially on C_o. By the lemma, $t^{-1}bt$ must lie in B.

Next, we show that $\tilde{T}$ normalizes T. Take $t_o \in T$. Then, by a similar computation in as the previous paragraph, $t^{-1}t_o t$ acts pointwise trivially on C_o, and stabilizes A_o as well. Again invoking the lemma, we conclude that this element lies in T.

The proofs of the other parts of the first assertion are postponed a little.

Now we prove that, as automorphisms of A_o, conjugation by $\tilde{T}$ stabilizes the set S of generators of $W = \mathcal{N}/T$. Take $s \in S$. Note that for any chamber C_1 adjacent to C_o, $t^{-1}C_1$ is necessarily a chamber in A_o adjacent to C_o, since $t^{-1}C_o = C_o$ and since chamber complex maps preserve adjacency. Also, t^{-1} permutes the vertices of C_o. Let v be any vertex of C_o fixed by the reflection s. Then $t^{-1}st$ fixes the vertex $t^{-1}v = t^{-1}sv$ of C_o. On the other hand, if v is the unique vertex of C_o *not* fixed by s, then $t^{-1}st$ maps the vertex $t^{-1}v$ of C_o to $t^{-1}sv$ (which is not a vertex of C_o). Thus, by the uniqueness lemma, $t^{-1}st$ must be the reflection through the facet $t^{-1}F$ where F is the facet of C_o fixed (pointwise) by s. That is, $\tilde{T}$ permutes the elements of S among themselves.

In particular, $\tilde{T}$ normalizes $W = \langle S\rangle$, as automorphisms of A_o. Note that if an automorphism ν of the building agrees on A_o with the action of an element of W, then ν necessarily preserves types on the whole building, by the lemma. Therefore, since $\tilde{T}$ normalizes T, $\tilde{T}$ normalizes $\mathcal{N}$.

Since $G = B\mathcal{N}B = BWB$, it follows that $\tilde{T}$ normalizes G. Given $g \in \tilde{G}$, by the assumed strong transitivity of G there is an element $h \in G$ so that $hgC_o = C_o$ and $hgA_o = A_o$. Thus, $hg \in \tilde{T}$. It follows that $\tilde{G} = \tilde{T}G = G\tilde{T}$.

In particular, at this point we obtain the remainder of the first point in the theorem, asserting that $\tilde{B} = \tilde{T}B$ and $\tilde{\mathcal{N}} = \tilde{T}\mathcal{N}$.

Granting the previous, the fact that $\tilde{\mathcal{N}}/T$ is a semi-direct product of $\Omega = \tilde{T}/T$ and W is clear. Likewise clear, then, is the fact that

$$\sigma BwB = B\sigma wB = B(\sigma w\sigma^{-1})B\sigma$$

since $\tilde{T}$ normalizes B. As in the discussion of strict BN-pairs (5.1) and (5.2), the cosets $\sigma B = B\sigma$ are well-defined.

Last, we address the finite index assertions. If two elements t_1, t_2 of $\tilde{T}$ have the same effect *pointwise* on C_o, then $t_1t_2^{-1}$ is trivial *pointwise* on C_o. By the lemma above, $t_1t_2^{-1}$ preserves types, so must lie in $T = \tilde{T} \cap G$. Thus, the natural map

$$\tilde{T}/T \to \{\text{ permutations of vertices of } C_o \text{ }\}$$

is an *injection*. Since S is finite and the vertices of C_o are in bijection with S, this permutation group is finite. Hence, $\tilde{T}/T$ is finite, as is $\tilde{G}/G$ since $\tilde{G} = \tilde{T}G$. ♣

5.6 The spherical case

Beyond the completely general results above much more can be said in case the building is *spherical*, that is, the apartments are *finite* complexes.

In the spherical case, we introduce **parabolic subgroups** of a group acting strongly transitively, **opposite parabolics**, and **Levi components** of parabolic subgroups. These are all conveniently defined in terms of the geometry of the building. We also can describe **associate parabolics** in such terms.

For example, we have shown (4.6) that there is a unique apartment system, which is therefore unavoidably maximal. In more detail, we have shown that any apartment is the *convex hull* of any two *antipodal chambers* within it, in the combinatorial sense that every other chamber in the apartment is in *some* minimal gallery connecting the two antipodal chambers, and every chamber occurring in such a minimal gallery is in that apartment.

Let X be a thick spherical building on which a group G acts by label-preserving simplicial complex automorphisms. Suppose that it is *strongly transitive*, that is, is transitive on pairs (C, A) where C is a chamber contained in an apartment A.

Since the apartment system is maximal, as observed earlier (5.2) it follows automatically that apartments are mapped to apartments by simplicial complex automorphisms.

Fix a chamber C in an apartment A, and identify A with a (finite) Coxeter complex $\Sigma(W, S)$ in such manner that $C = \langle\emptyset\rangle$ and S is the collection of reflections in the facets of C, as in (4.3), (3.4).

Let $\mathcal{N}$ be the stabilizer of A in G. Rather than using the letter B for the stabilizer of C, in the spherical case we let P be the stabilizer of C in G. And we call P the **minimal parabolic subgroup** associated to the chamber C. Instead of the symbol T for $\mathcal{N} \cap P$ as above, we now write $M = \mathcal{N} \cap P$. And then $W = \mathcal{N}/M$. We call M the **Levi component** M of P corresponding to choice of apartment A. And the Coxeter group W is called the **(spherical) Weyl group** associated to choice of C and A.

Let C^{opp} be the *antipodal chamber* to C in the apartment A (4.6). The stabilizer P^{opp} of C^{opp} is the **opposite parabolic** to P, with respect to the apartment A. That is, of all the chambers in X which are the maximal gallery distance from C, we have specified C^{opp} by telling in which apartment containing C it lies. As remarked just above, we proved earlier that, in effect, the collection of chambers at maximal gallery distance from C is naturally in bijection with the collection of apartments containing C, in (4.6).

Proposition: The Levi component $M = \mathcal{N} \cap P$ is none other than $P \cap P^{\text{opp}}$. The collection of all Levi components in the minimal parabolic P is acted-upon transitively by the conjugation action of P upon itself. Equivalently, the minimal parabolic acts transitively by conjugation on the set of parabolic subgroups *opposite* to it. Equivalently, P acts transitively on the set of all chambers antipodal (in any apartment) to the chamber stabilized by P.

Proof: It is clear that $M = \mathcal{N} \cap P$ fixes C^{opp} since it fixes the whole apartment A in which this chamber lies. Thus $M \subset P \cap P^{\text{opp}}$. On the other hand, if $g \in G$ fixes both C and C^{opp}, then it certainly *stabilizes* the collection of minimal galleries from C to C^{opp}. Keep in mind that *every* minimal gallery between these chambers lies in A, by the combinatorial convexity of apartments in general proven above (4.5). Further, by the Uniqueness Lemma (3.2), since g fixes C and maps to the thin chamber complex A, it must be that g is the identity on any such gallery. Thus, g is the identity map on all of A.

The second assertion is a covert version of the strong transitivity. Indeed, by definition (5.2) of the strong transitivity of G on X, P is transitive on apartments B containing C. In each such apartment there is a *unique* chamber C_B^{opp} antipodal to C with stabilizer P_B^{opp}. The corresponding Levi component of P is

$$M_B = P \cap P_B^{\text{opp}}$$

But the transitivity and the uniqueness of antipodal chamber (to C) within a given apartment (4.6) prove that P is transitive on such chambers. Thus, P acts transitively by conjugation on the opposite parabolics P_B^{opp}, and therefore transitively on the Levi components M_B. ♣

Remarks: By symmetry, the subgroup $M = P \cap P^{\text{opp}}$ is also the Levi component of P^{opp} corresponding to the apartment A, and M certainly stabilizes the opposite chamber C^{opp}.

Corollary: The Weyl group $W^{\text{opp}} = \mathcal{N}/(\mathcal{N} \cap P^{\text{opp}})$ can be naturally identified with the Weyl group $W = \mathcal{N}/(\mathcal{N} \cap P)$.

Proof: We have seen that

$$\mathcal{N} \cap P = P \cap P^{\text{opp}}$$

which gives a symmetrical expression for M. ♣

Now we define more general **parabolic subgroups** and their **opposite parabolic**, as well as **Levi components**. First, any subgroup of G fixing some simplex σ in X is said to be a **parabolic subgroup**. Any such group certainly contains the fixer of a chamber of which σ is a face. Thus, by whatever definition, we may be sure that *parabolic subgroups always contain minimal parabolic subgroups, which are fixers of chambers.*

From the general results (5.3), we know that any subgroup Q containing the minimal parabolic P is of the form

$$Q = P_T = \bigsqcup_{w \in \langle T \rangle} PwP$$

where T is a subset of S and $\langle T \rangle$ is the subgroup of W generated by T. In this notation we have $P = P_\emptyset$.

With regard to the choice A of apartment containing C, and corresponding opposite P^{opp}, define the **opposite** parabolic Q^{opp} to Q by

$$Q^{\mathrm{opp}} = \bigsqcup_{w \in \langle T \rangle} P^{\mathrm{opp}} w P^{\mathrm{opp}}$$

The **Levi component** M_Q of such a parabolic subgroup Q, corresponding to the apartment A is

$$M_Q = Q \cap Q^{\mathrm{opp}}$$

Remarks: Of course, elements $w \in W$ must be replaced by representatives from $\mathcal{N}$ in the previous expression. The complication is that we have $W = \mathcal{N}/M$ where $M = \mathcal{N}/(\mathcal{N} \cap P)$. But there is no difficulty, since the corollary just above shows that

$$\mathcal{N} \cap P = P \cap P^{\mathrm{opp}} = \mathcal{N} \cap P^{\mathrm{opp}}$$

Remarks: Since these opposite parabolics P_T^{opp} contain $P^{\mathrm{opp}} = P_\emptyset^{\mathrm{opp}}$, they certainly *are* parabolic subgroups in our present sense.

The following easy proposition displays opposite parabolics in a manner conforming more to our earlier discussion:

Proposition: Let w_o be the (unique) *longest* element in the finite Coxeter group W. Then $w_o C = C^{\mathrm{opp}}$ and $P^{\mathrm{opp}} = w_o P w_o^{-1}$. We have $w_o^2 = 1 \in W$. Thus, in general, for a parabolic P_T with $T \subset S$, we have

$$P_T^{\mathrm{opp}} = w_o(\bigsqcup_{w \in w_o^{-1} \langle T \rangle w_o} PwP) w_o^{-1}$$

Proof: From discussion of Coxeter complexes in general (3.4) we know that the gallery distance from $C = \{1\}$ to any other chamber $\{w\}$ is the length of w. Thus, it must be that $C^{\mathrm{opp}} = \{w_o\}$. That is,

$$C^{\mathrm{opp}} = \{w_o\} = w_o\{1\} = w_o C$$

(We already showed, in discussion of finite Coxeter groups (1.5), that there is a unique longest element w_o. The present discussion appears to give another proof.)

Because w_o gives a simplicial automorphism of A, a minimal gallery γ from C to C^{opp} is mapped to a minimal gallery $w_o\gamma$ from $w_o C = C^{\mathrm{opp}}$ to $(w_o)^2 C$.

Since C is the unique chamber antipodal (in A) to C^{opp}, and since gallery lengths are preserved by such maps, necessarily $(w_o)^2C = C$. Thus,

$$\{1\} = C = (w_0)^2C = (w_o)^2\{1\} = \{w_o^2\}$$

which implies that $w_o^2 = 1 \in W$.

The last assertion is a direct computation on the *Bruhat cells* $P^{\text{opp}}wP^{\text{opp}}$:

$$P^{\text{opp}}wP^{\text{opp}} = w_oPw_o^{-1}ww_oPw_o^{-1} = w_o(P(w_o^{-1}ww_o)P)w_o^{-1}$$

giving the desired conclusion. ♣

Remarks: As S was identified with reflections in the facets of C, the set $w_o^{-1}Sw_o^{-1}$ may be identified with reflections in the facets of the opposite chamber $C^{\text{opp}} = w_oC$. Thus, while the Coxeter group W remains the same, the *system* (W, S) should be replaced by (W, w_oSw_o) when C is replaced by $C^{\text{opp}} = w_oC$.

Corollary: Let w_o be the longest element in a spherical Coxeter group W. The map $w \to w_oww_o^{-1}$ gives an automorphism of W of order 2 which stabilizes the generating set S.

Proof: We already saw that $w_o^2 = 1$. The previous little result shows that, among other things, for every $s \in S$ the conjugate $w_o\langle s\rangle w_o^{-1}$ is again a special subgroup of W. Thus, by counting considerations, it must be of the form $\langle s'\rangle$. That is, $w_osw_o^{-1} = s'$, showing that we have an automorphism of S. ♣

Remarks: All *minimal parabolics* are conjugate to each other (from the transitivity of G on chambers), so in particular a minimal parabolic P is conjugate in G to its opposite P^{opp}, with respect to any choice of apartment (equivalently, Levi component). By contrast, there is no reason to expect that *non-minimal* parabolics be conjugate to their opposites, although necessarily all opposites of a given parabolic are conjugate to each other.

In certain situations involving spherical BN-pairs, minimal parabolics are also called **Borel subgroups**.

5.7 Buildings from BN-pairs

Under very mild hypotheses, all BN-pairs arise from group actions upon buildings, and in an essentially unique manner. (The argument *does not* use any result about a BN-pair presuming that it comes from a building).

Let B, $\mathcal{N}$ be a BN-pair in a group G. We *assume* that the generating set S for the Coxeter group $W = \mathcal{N}/(\mathcal{N} \cap B)$ is *finite*. (Note that this does *not* imply that W is finite).

For purposes of this section, a (proper) **parabolic subgroup** of G is any proper subgroup of G which contains some conjugate gBg^{-1} of B (by $g \in G$). The collection of all proper parabolics can be made into a poset X by taking the *reverse of inclusion* as the face relation. This poset will be shown to be a building giving rise to the given BN-pair (5.2).

The collection of apartments is described as follows: first, let

$$A = \{wPw^{-1} : P \text{ is a special subgroup } , w \in \mathcal{N}\}$$

be the (alleged) apartment containing the (alleged) chamber B, and then for any $g \in G$ let

$$gA = \{gwPw^{-1}g^{-1} : P \text{ is a special subgroup } , w \in \mathcal{N}\}$$

also be declared to be an apartment.

The action of G upon X is declared to be by *conjugation* of subgroups.

Theorem: Let $B, \mathcal{N}$ be a BN-pair. Let Ξ be the poset of proper parabolic subgroups of G, with inclusion *reversed*, as just above, and with the indicated apartment system. Then X is a simplicial complex which is, in fact, a thick building X upon which G acts in a label-preserving manner, with B occuring as the stabilizer of a chamber inside an apartment stabilized by $\mathcal{N}$.

Proof: The proof is made somewhat easier by replacing X by an apparently simpler (but poset-isomorphic) object, described as follows:

For present purposes, a *special* subgroup of G is a *proper* subgroup P of G containing B. A **special subset** of G is a subset of the form gP for P a special subgroup and $g \in G$. The poset Y obtained by ordering all special subsets with the *reverse* of containment is our candidate for the building.

The action of G upon special subsets is taken to be left multiplication.

Proposition: The poset Y of all special subsets of G (with inclusion reversed) is isomorphic (as poset) to the poset X of all proper parabolic subsets (with inclusion reversed), by the map

$$f : gP \to gPg^{-1}$$

Further, this map respects the action of G upon X and Y.

Proof: Each special subgroup is its own normalizer in G, and no two of them are conjugate (5.3). This implies that the indicated map is well-defined, and is an injection. Thus, it is certainly a bijection, since its surjectivity follows from its well-defined-ness. Further, if $gP \subset hQ$ for special subgroups P, Q, then $(h^{-1}g)P \subset Q$, so

$$h^{-1}g = h^{-1}g \cdot e \subset h^{-1}g \cdot P \subset Q$$

and $P \subset Q$. Therefore,

$$P \subset Q = (g^{-1}h)Q(g^{-1}h)^{-1})$$

and

$$gPg^{-1} \subset hQh^{-1}$$

Thus, the poset structure is preserved by the map.

Finally, for $g, h \in G$ is it clear that

$$f(g(hP)) = f((gh)P) = (gh)P(gh)^{-1} = g(hPh^{-1})g^{-1} = g(f(hP))$$

so the action of G is preserved by the map. This proves the proposition. ♣

Now we return to the proof of the theorem, at each moment using whichever model of the purported building is more convenient. The candidate for the apartment system in Y is as follows, translating from the corresponding subcomplex of X: First, the collection

$$A = \{wB : w \in \mathcal{N}\}$$

is declared to be an apartment. And for every $g \in G$ we also declare

$$gA = \{gwB : w \in \mathcal{N}\}$$

to be an apartment.

It is necessary to prove that X (or, equivalently, Y) is a chamber complex. To do this, it suffices (3.1) to show that any two elements (alleged simplices) x, y have a unique greatest lower bound, and that for each $x \in X$ the sub-poset

$$Y_{\leq x} = \{y \in X : y \leq x\}$$

is *simplex-like* (meaning that it is isomorphic to the set of subsets of some finite set).

Let S_1, S_2 be two subsets of S, let

$$P_i = \bigsqcup_{w \in \langle S_i \rangle} BwB$$

let g_1, g_2 be in G, and suppose that two special subsets g_1P_1 and g_2P_2 are contained in a special subset gP (strictly smaller than G). By left-multiplying by g^{-1}, we may suppose without loss of generality that $g = 1$.

Then $g_iP_i \subset P$ for $i = 1, 2$ and

$$g_i = g_i \cdot 1 \in g_i \cdot P_i \in P$$

Thus, also, $P_i \subset P$. This is true for *any* special subgroup P with $g_iP_i \subset P$, so we can take the intersection of all special subgroups containing both g_1P_1 and g_2P_2 to obtain the greatest lower bound (with inclusion reversed).

Next, given a special subset gP, we classify the special subsets $g'P'$ containing gP. By left multiplying by g^{-1}, we may assume without loss of generality $g = 1$. Then $P \subset g'P'$ implies ${g'}^{-1}P \subset P'$, so actually $g' \in P'$. Thus, simply, $P \subset P'$. Let (5.1), (5.3) S_o, S' be the subsets of S so that

$$P = \bigsqcup_{w \in \langle S_o \rangle} BwB$$

$$P' = \bigsqcup_{w \in \langle S' \rangle} BwB$$

That is, the collection of all such P' is in bijection with

$$\{S' : S_o \subset S' \subset S \quad \text{but} \quad S' \neq S\}$$

Invoking the finiteness of S, this collection is finite. Thus, we have proven that $X \approx Y$ are simplicial complexes.

Now we begin to prove that $X \approx Y$ is a thick building, upon which G acts preserving labels, with B and $\mathcal{N}$ arising as the associated BN-pair.

To prove that $X \approx Y$ is a chamber complex, it will suffice to prove that any two chambers lie in a common apartment, and that each apartment is a chamber complex. To prove the latter, it suffices to prove that each apartment is a Coxeter complex (3.4). Let $\Sigma = \Sigma(W, S)$ be the Coxeter complex associated to the Coxeter system (W, S) (3.4), by definition being the poset consisting of all subsets $w\langle S' \rangle$ of W with $S' \subset S$ and $w \in W$, with inclusion reversed. Consider the map

$$f : w\langle S' \rangle \to wBW_{S'}B$$

from Σ to the apartment A, where $W_{S'} = \langle S' \rangle$ is as usual the subgroup of W generated by S'.

Noting that $W_{S'}B$ is a well-defined subset of the sub*group* $BW_{S'}B$, the map f is well-defined, since replacing w by ww' with $w' \in W_{S'}$ has the effect that

$$f(ww'W_{S'}) = (ww')BW_{S'}B = w(w'BW_{S'}B) = w(BW_{S'}B) = f(wW_{S'})$$

For emphasis, the key point here is that for any subset S' of S the subset $BW_{S'}B$ is a *subgroup* (5.3). The map is surjective, just from the definitions. To prove injectivity, suppose $f(w_1 W_{S_1}) = f(w_2 W_{S_2})$. By left multiplying by w_2^{-1}, we may suppose without loss of generality that $w_2 = 1$. Then we have

$$w_1 BW_{S_1}B = BW_{S_2}B$$

Since the sets $BW_{S_i}B$ are *groups*, we conclude that $w_1 \in BW_{S_2}B$, and that $BW_{S_1}B = BW_{S_2}W$. Since $BW_{S_1}B \subset BW_{S_2}W$ implies $S_1 \subset S_2$ (5.3), we have $S_1 = S_2$. This proves injectivity.

Thus, the map f gives a poset isomorphism from the Coxeter complex Σ to the alleged apartment A. In particular, A (and all the images gA) are *thin chamber complexes*.

For one of the building axioms, given two simplices g_1P_1, g_2P_2 in Y, we must find an apartment containing both. We certainly may restrict our attention to *chambers*, since by now we know that the apartments really are simplicial complexes (and in particular contain all faces of all their simplices). So $P_1 = P_2 = B$, and without loss of generality we may suppose that one of the chambers is B itself. Let the other chamber be gB. Write $g = bwb'$ in a Bruhat decomposition, where $b, b' \in B$ and $w \in W$. Then

$$gB = (bwb')B = b(wB) \in bA$$

Thus, $gB \in bA$, and certainly $B = bB \in A$, so the apartment bA contains the two given chambers.

Next, we prove strong transitivity. The transitivity of G on apartments gA in Y is clear. To prove *strong* transitivity, it suffices to prove that the stabilizer of A is transitive on chambers in A. Certainly $\mathcal{N}$ is *contained* in the stabilizer of A, and since

$$W = \mathcal{N}/(\mathcal{N} \cap B)$$

it is likewise clear that $\mathcal{N}$ is transitive on chambers in A. This proves that G is strongly transitive on Y.

The labelling on Y uses the unique expression of every special subgroup P in the form

$$P = BW_{S_P}B$$

for some subset S_P of S. Then use the labelling

$$\lambda(gP) = S - S_P$$

where the subtraction indicates set complement. (The complement is used to comply with conventions used elsewhere!) If this labelling is well-defined it is certainly preserved by the action of G. As usual, if $gP = hQ$ for special subgroups P, Q, then left multiply by h^{-1} so suppose that $h = 1$ without loss of generality. Then $g = g \cdot 1 \in gP = Q$ implies that $g \in Q$, and then $P = Q$. This proves well-definedness of this labelling.

Now we verify that if two apartments have a common chamber, then there is a simplicial isomorphism of the two fixing their intersection pointwise. Invoking strong transitivity, we may assume that the common chamber is B, that one of the two apartments is A, that the other is bA with $b \in B$, and thus that B itself is a chamber common to the two apartments. Consider the map

$$f : A \to bA$$

defined by $f(wP) = bwP$.

It remains to show that if $wP \subset bA$ (in addition to $wP \in A$) then $f(wP) = wP$. That is, we must show that $wP \in bA$ implies that $bwP = wP$. Suppose that $wP = bw'Q$ for a special subgroup Q, and for some $w' \in W$. Then

$$w^{-1}bw' = w^{-1}bw' \cdot 1 \in w^{-1}bw' \cdot Q = P$$

and $Q = P$. Then

$$BwP = B \cdot wP = B \cdot bw'P = Bw'P$$

Let $P = BW_{S'}B$ where $W_{S'}$ is the subgroup of S generated by a subset S' of S. We have

$$w' \in BwP \subset \bigcup_{w_1 \in S'} BwBw_1B$$

For fixed $w_1 \in W_{S'}$, write

$$w_1 = s_1 \ldots s_n$$

with $s_1, \ldots, s_n \in S'$. By iterated application of the cell multiplication rules (5.1), we have

$$BwBw_1B \subset \bigcup_{\varepsilon_1, ldots, \varepsilon_n} Bws_1^{\varepsilon_1} \ldots s_n^{\varepsilon_n} B$$

where the ε_i vary over $\{0, 1\}$. In particular, we find that w' lies in some Bww_2B for $w_2 \in W_{S'}$. By the Bruhat decomposition for G, the double cosets $Bw'B$ and Bww_2B are *disjoint* unless $w' = ww_2$. In the latter case, $w^{-1}w' \in W_{S'}$ and $w^{-1}w'B \subset P$, and, thus $w'P = wP$.

Then

$$f(wP) = bwP = w'P = wP$$

as desired, proving that f fixes $A \cap bA$, as required by the building axioms.

Last, we verify the *thickness* of the building Y. That is, given a codimension-one simplex (facet) F we must find at least 3 chambers of which it is a facet. Invoking the transitivity of G on Y, it suffices to consider a facet F of the chamber $B = 1 \cdot B$. Every such facet is of the form

$$P_s = B \sqcup BsB$$

for some $s \in S$. In addition to B itself, we must find two other special subsets gB so that $gB \subset P_s$ (recalling that the partial ordering is the *reverse* of containment). One of the two is obvious: the coset sB. To see what's going on generally, the point is that we want the coset space

$$P_s/B = (B \sqcup BsB)/B$$

to have *three or more* elements. Generally for subgroups M, N of a group H and for $h \in H$ we have a natural bijection

$$MhN/N \approx M/(M \cap hNh^{-1})$$

by the map

$$xN \to xh(M \cap hNh^{-1})$$

as is straightforward to check. Thus,

$$BsB/B \approx B/(sBs \cap B)$$

Now one of the axioms for a BN-pair is that $sBs \neq B$. Thus,

$$[B : sBs \cap B] \geq 2$$

and we have the desired thickness of Y.

♣

6. Hecke Algebras

- Generic algebras
- Strict Iwahori-Hecke algebras
- Generalized Iwahori-Hecke algebras

In various classical settings, in in some not-so-classical ones, there are rings of operators called *Hecke algebras* which play important technical roles.

6.1 Generic algebras

Let (W, S) be a Coxeter system, and fix a commutative ring R. We consider S-tuples of pairs (a_s, b_s) of elements of R, subject only to the requirement that if $s_1 = ws_2w^{-1}$ for $w \in W$ and $s_1, s_2 \in S$, then $a_{s_1} = a_{s_2}$ and $b_{s_1} = b_{s_2}$. We will refer to the constants a_s, b_s as *structure constants*. Let $\mathcal{A}$ be a free R-module with R-basis $\{T_w : w \in W\}$.

Theorem: Given a Coxeter system (W, S) and structure constants a_s, b_s, there is exactly one associative algebra structure on $\mathcal{A}$ so that

$$T_s T_w = T_{sw} \qquad \text{if} \quad \ell(sw) > \ell(w)$$

$$T_s^2 = a_s T_s + b_s T_1 \qquad \forall s \in S$$

and with the requirement that T_1 is the identity in $\mathcal{A}$. With this associative algebra structure, we also have

$$T_s T_w = a_s T_w + b_s T_{sw} \qquad \text{if} \quad \ell(sw) < \ell(w)$$

Further, we have the right-handed version of these identities:

$$T_w T_s = T_{ws} \qquad \text{if} \quad \ell(ws) > \ell(w)$$

$$T_w T_s = a_s T_w + b_s T_{ws} \qquad \text{if} \quad \ell(ws) < \ell(w)$$

Granting the theorem, for given data we define the **generic algebra**

$$\mathcal{A} = \mathcal{A}(W, S, \{(a_s, b_s) : s \in S\})$$

to be the associative R-algebra determined according to the theorem.

Remarks: If all $a_s = 0$ and $b_s = 1$ then the associated generic algebra is the *group algebra* of the group W over the ring R. Recall that this is the free R-module on generators $[w]$ for $w \in W$, and with multiplication

$$(r[w])(r'[w']) = (rr')[ww']$$

for $r, r' \in R$ and $w, w' \in W$. We will not attempt to exploit the positive attributes of such rings here.

Remarks: When (W, S) is *affine*, as illustrated in (2.2) or generally below in (12.4) and (13.6), certain less obvious choices of structure constants yield the *Iwahori-Hecke algebra* in p-adic groups. Most often, this is

$$a_s = q - 1 \qquad b_s = q$$

where q is the residue field of the relevant discrete valuation ring, etc.

Proof: First, we see that the 'right-handed' version of the statements follows from the 'left-handed' ones. Suppose that $\ell(wt) > \ell(w)$ for $w \in W$ and $t \in S$. Take any $s \in S$ so that $\ell(sw) < \ell(w)$. We certainly have

$$\ell(w) = \ell((sw)t) > \ell(sw)$$

Then we have

$$T_w T_t = T_s T_{sw} T_t = T_s T_{swt} = T_{wt}$$

where the first equality follows from $\ell(w) = \ell(ssw) > \ell(sw)$, where the second follows by induction on length, and the third equality follows from $\ell(sswt) > \ell(swt)$. This gives the desired result. If $\ell(wt) < \ell(w)$, then by the result just proven $T_{wt} T_t = T_w$. Multiplying both sides by T_t on the right yields

$$T_w T_t = T_{wt} T_t^2 = T_{wt}(a_t T_t + b_t T_1) = a_t T_{wt} T_t + b_t T_{wt} =$$

$$= a_t T_w + b_t T_{wt}$$

where we computed T_t^2 by the defining relation. Thus, the right-handed versions do follow from the left-handed ones.

Next, suppose that $\ell(sw) < \ell(w)$ and prove that

$$T_s T_w = a_s T_w + b_s T_{sw}$$

If $\ell(w) = 1$, then $w = s$, and the desired equality is just the assumed equality

$$T_s^2 = a_s T_s + b_s T_1$$

Generally, $\ell(s(sw)) > \ell(sw)$, so $T_s T_{sw} = T_w$. Then

$$T_s T_w = T_s^2 T_{sw} = (a_s T_s + b_s T_1) T_{sw} =$$

$$= a_s T_s T_{sw} + b_s T_{sw} = a_s T_w + b_s T_{sw}$$

as asserted. Thus, the more general multiplication rule applicable when $\ell(sw) < \ell(w)$ follows from the rule for $\ell(sw) > \ell(w)$ and from the formula for T_s^2.

Uniqueness is also easy. If $w = s_1 \ldots s_n$ is reduced, then

$$T_w = T_{s_1} \ldots T_{s_n}$$

Therefore, $\mathcal{A}$ is generated as an R-algebra by the T_s (for $s \in S$ and T_1. Then the relations of the theorem allow us to write down the rule for multiplication of any two elements T_{w_1} and T_{w_2}. There is no further choice to be made, so we have a unique algebra structure satisfying the relations indicated in the theorem.

Now we prove existence of this associative algebra, for given data. Let $\mathcal{A}$ also denote the free R-module on elements T_w for $w \in W$. In the ring $\mathcal{E} = \mathrm{End}_R(\mathcal{A})$ we have *left multiplications* λ_s and *right multiplications* ρ_s for $s \in S$ given by

$$\begin{aligned} \lambda_s(T_w) &= T_{sw} \quad \text{for} \quad \ell(sw) > \ell(w) \\ \rho_s(T_w) &= T_{ws} \quad \text{for} \quad \ell(ws) > \ell(w) \\ \lambda_s(T_w) &= a_s T_w + b_s T_{sw} \quad \text{for} \quad \ell(sw) > \ell(w) \\ \rho_s(T_w) &= a_s T_w + b_s T_{ws} \quad \text{for} \quad \ell(ws) > \ell(w) \end{aligned}$$

We grant for the moment that the λ_s commute with the ρ_t: we will prove this below. Let Λ be the subalgebra of $\mathcal{E}$ generated by the λ_s. Let $\phi : \Lambda \to \mathcal{A}$ by $\phi(\alpha) = \alpha(T_1)$. Thus, for example, $\phi(1) = T_1$ and, for all $s \in S$, $\phi(\lambda_s) = T_s$.

Certainly ϕ is a surjective R-module map, since for every reduced expression $w = s_1 \ldots s_n$ we have

$$\phi(\lambda_{s_1} \ldots \lambda_{s_n}) = (\lambda_{s_1} \ldots \lambda_{s_n})(1) =$$
$$= \lambda s_1 \ldots \lambda_{s_{n-1}} T_{s_n} = \lambda s_1 \ldots \lambda_{s_{n-2}} T_{s_{n-1} s_n} =$$
$$\ldots = T_{s_1 \ldots s_n} = T_w$$

To prove injectivity of ϕ, suppose $\phi(\alpha) = 0$. We will prove, by induction on $\ell(w)$, that $\alpha(T_w) = 0$ for all $w \in W$. By definition, $\phi(\alpha) = 0$ means $\alpha(T_1) = 0$. Now suppose $\ell(w) > 0$. Then there is $t \in S$ so that $\ell(wt) < \ell(w)$. We are assuming that we already know that ρ_t commutes with Λ, so

$$\alpha(T_w) = \alpha(T_{(wt)t}) = \alpha \rho_t T_{wt} = \rho_t \alpha T_{wt} = 0$$

by induction on length.

Thus, Λ is a free R-module with basis $\{\lambda_w : w \in W\}$. We note that this R-module isomorphisms also implies that $\lambda_w = \lambda_{s_1} \ldots \lambda_{s_n}$ for any reduced expression $w = s_1 \ldots s_n$. The natural R-algebra structure on Λ can be 'transferred' to $\mathcal{A}$, leaving only the checking of the relations.

To check the relations: suppose that $\ell(sw) > \ell(w)$. For a reduced expression $w = s_1 \ldots s_n$, the expression $ss_1 \ldots s_n$ is a reduced expression for sw. Thus,

$$\lambda_s \lambda_w = \lambda_s \lambda_{s_1} \ldots \lambda_{s_n} = \lambda_{sw}$$

That is, we have the desired relation $\lambda_s \lambda_w = \lambda_{sw}$.

We check the other relation $\lambda_s^2 = a_s \lambda_s + b_s \lambda_1$ by evaluating at $T_w \in \mathcal{A}$. For $\ell(sw) > \ell(w)$,

$$\lambda_s^2(T_w) = \lambda_s(\lambda_s T_w) = \lambda_s(T_{sw}) = a_s T_{sw} + b_s T_w =$$
$$= a_s \lambda_s T_w + b_s \lambda_1 T_w = (a_s \lambda_s + b_s \lambda_1) T_w$$

If $\ell(sw) < \ell(w)$, then

$$\lambda_s^2(T_w) = \lambda_s(\lambda_s T_w) = \lambda_s(a_s T_w + b_s T_{sw}) =$$
$$= a_s \lambda_s T_w + b_s T_s T_{sw} = a_s \lambda_s T_w + b_s \lambda_1 T_w = (a_s \lambda_s + b_s \lambda_1) T_w$$

This proves that $\lambda_s^2 = a_s \lambda_s + b_s \lambda_1$, as desired.

The argument is complete except for the fact that the left and right multiplication operators defined above commute with each other. To prepare to prove this, we need to carry out a little exercise on Coxeter groups:

Proposition: Let (W, S) be a Coxeter system. Let $w \in W$ and $s, t \in S$. If both $\ell(swt) = \ell(w)$ and $\ell(sw) = \ell(wt)$, then $swt = w$ (and $s = wtw^{-1}$). In particular, $a_s = a_t$ and $b_s = b_t$, since s and t are conjugate.

Proof: Let $w = s_1 \dots s_n$ be a reduced expression. On one hand, for $\ell(sw) > \ell(w)$,

$$\ell(w) = \ell(s(wt)) < \ell(sw)$$

so the Exchange Condition (1.7) applies: there is $v \in W$ so that $sw = vt$ and so that either $v = ss_1 \dots \hat{s_i} \dots s_n$ or $v = w$. But $v = ss_1 \dots \hat{s_i} \dots s_n$ is not possible, since this would imply that

$$\ell(wt) = \ell(s_1 \dots \hat{s_i} \dots s_n) < \ell(w)$$

contradicting the present hypothesis

$$\ell(wt) = \ell(sw) > \ell(w)$$

On the other hand, for $\ell(sw) < \ell(w) = \ell(s(sw))$, the hypotheses of this claim are met by sw in place of w, so the previous argument applies. We conclude that $s(sw) = (sw)t$, which gives $w = swt$, as desired. This proves the proposition. ♣

Now we can get to the commutativity of the operators:

Lemma: For all $s, t \in S$, the operators $\lambda_s, \rho_t \in \mathcal{E}$ commute.

Proof: We will prove that $\lambda_s \rho_t - \rho_t \lambda_s = 0$ by evaluating the left-hand side on T_w. There is a limited number of possibilities for the relative lengths of w, sw, wt, swt, and in each case the result follows by direct computation, although we need to use the *claim* in two of them:

If $\ell(w) < \ell(wt) = \ell(sw) < \ell(swt)$, then by the definitions of the operators λ_s, ρ_t we have

$$\lambda_s \rho_t T{+}_w = \lambda_s T_{wt} = T_{swt}$$

In the opposite case $\ell(w) > \ell(wt) = \ell(sw) > \ell(swt)$,

$$\lambda_s \rho_t T_w = \lambda_s(a_t T_w + b_t T_{wt}) = a_t(a_s T_w + b_s T_{sw}) + +b_t(a_s T_{wt} + b_s T_{swt})$$

which, by rearranging and reversing the argument with s and t and left and right interchanged, is

$$= a_s(a_t T_w + b_t T_{wt}) + +b_s(a_t T_{sw} + b_t T_{swt}) = \rho_t \lambda_s T_w$$

In the case that $\ell(wt) = \ell(sw) < \ell(swt) = \ell(w)$, we invoke the *claim* just above: we have $a_s = a_t$ and $b_s = b_t$, and $sw = wt$. Then we compute directly:

$$\lambda_s \rho_t T_w = \lambda_s(a_t T_w + b_t T_{wt}) =$$

$$= a_t(a_s T_w + b_s T_{sw}) + b_t T_{swt} =$$

$$= a_s(a_t T_w + b_t T_{wt}) + b_s T_{swt} = \rho_t(a_s T_w + b_s T_{sw}) =$$

$$= \rho_t \lambda_s T_w$$

as desired.

In the case that $\ell(wt) < \ell(w) = \ell(swt) < \ell(sw)$,

$$\lambda_s \rho_t T_w = \lambda_s(a_t T_w + b_t T_{wt}) = a_t T_{sw} + b_t T_{swt} =$$

$$= \rho_t(\lambda_s T_w)$$

In the case opposite to the previous one, that is, that $\ell(sw) < \ell(w) = \ell(swt) < \ell(wt)$, a symmetrical argument applies.

In the case that $\ell(w) = \ell(swt) < \ell(sw) = \ell(wt)$, we again invoke the *claim* above, so that we have $a_s = a_t$ and $b_s = b_t$, and also $sw = wt$. Then

$$\lambda_s \rho_t T_w = \lambda_s T_{wt} = a_s T_{wt} + b_s T_{swt} =$$

$$= a_t T_{sw} + b_t T_{swt} = \rho_t T_{sw} = \rho_t \lambda_s T_w$$

This finishes the proof of commutativity, and thus of the theorem on generic algebras. ♣

6.2 Iwahori-Hecke algebras

This section demonstrates that the Iwahori-Hecke algebras do indeed qualify as generic algebras in the sense above. Surprisingly, the whole line of argument only depends upon a *local finiteness* property of the building.

Let G be a group acting strongly transitively on a thick building X, preserving a labeling, all as in (5.2). (Again, the strong transitivity means that G is transitive upon pairs $C \subset A$ where C is a chamber in an apartment A in the implicitly given apartment system.) Let (W, S) be the Coxeter system associated to the apartments: each apartment is isomorphic to the Coxeter complex of this pair (W, S). Let B be the stabilizer of C. We assume always that S is finite.

The **local finiteness** hypothesis is that *we assume that for all $s \in S$ the cardinality*

$$q_s = \mathrm{card}(BsB/B) = \mathrm{card}(B\backslash BsB)$$

is finite. Recall that the subgroup of G stabilizing the facet F_s of C of type $\{s\}$ for $s \in S$ is none other than

$$P = P_s = B\langle s\rangle B = B \sqcup BsB$$

The subgroup B is the subgroup of P additionally stabilizing C, so BsB is the subset of $B\langle s\rangle B$ mapping C to *another* chamber s-adjacent to C (that is, with common facet F_s of type $\{s\}$). Therefore, BsB/B is in bijection with the set of chambers s-adjacent to C (other than C itself), by $g \to gC$.

That is, our *local finiteness* hypothesis is that *every facet is the facet of only finitely-many chambers*. Equivalently, since S is finite we could assume that *each chamber is adjacent to only finitely-many other chambers*.

Fix a field k of characteristic zero. Let

$$\mathcal{H} = \mathcal{H}_k(G, B)$$

be the *'Iwahori-Hecke algebra'* in G over the field k, that is, the collection of left and right B-invariant k-valued functions on G which are supported on finitely-many cosets Bg in G. As usual, the left and right B-invariance is the requirement that $f(b_1 g b_2) = f(g)$ for all $g \in G$ and $b_1, b_2 \in B$.

We have a *convolution product* on $\mathcal{H}$, given by

$$(f * \phi)(g) = \sum_{h \in B\backslash G} f(gh^{-1})\phi(h)$$

The hypothesis that ϕ is supported on finitely-many cosets Bx implies that the sum in the previous expression is *finite.* Since ϕ is left B-invariant and f is right B-invariant the summands are constant on cosets Bg, so summing over $B\backslash G$ makes sense. *Nevertheless, we must provide proof that the product is again in* $\mathcal{H}$. We do this in the course of the theorem.

Generally, let ch_E be the characteristic function of a subset E of G. By the Bruhat-Tits decomposition, *if indeed they are in* $\mathcal{H}(G, B)$, the functions ch_{BwB} form an k-basis for $\mathcal{H}(G, B)$. This Hecke algebra is visibly a free k-module.

Theorem: Each BgB is indeed a finite union of cosets Bx, the algebra $\mathcal{H}$ is closed under convolution products, and we have

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = \mathrm{ch}_{BswB} \quad \text{for} \quad \ell(sw) > \ell(w)$$

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BsB} = a_s \mathrm{ch}_{BsB} + b_s \mathrm{ch}_B$$

for

$$a_s = q_s - 1 \ \text{ and } \quad b_s = q_s$$

That is, *these Iwahori-Hecke operators form a generic algebra with structure constants as indicated.* Further, for a reduced expression $w = s_1 \ldots s_n$ (that is, with $n = \ell(w)$ and all $s_i \in S$), we have

$$q_w = q_{s_1} \ldots q_{s_n}$$

Proof: We first prove that double cosets BwB are finite unions of cosets Bx at the same time that we study one of the requisite identities for the generic algebra structure. This also will prove that $\mathcal{H}$ is closed under convolution products.

We do induction on the length of $w \in W$. Take $s \in S$ so that $\ell(sw) > \ell(w)$. At $g \in G$ where $\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB}$ does not vanish, there is $h \in G$ so that $\mathrm{ch}_{BsB}(gh^{-1})\mathrm{ch}_{BwB}(h) \neq 0$. For such h, we have $gh^{-1} \in BsB$ and $h \in BwB$. Thus, by the Bruhat cell multiplication rules,

$$g = (gh^{-1})h \in BsB \cdot BwB = BswB$$

Since this convolution product is left and right B-invariant, we conclude that

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = c\, \mathrm{ch}_{BswB}$$

for some *positive rational number* $c = c(s,w)$.

Let us compute the constant $c = c(s,w)$, by summing the previous equality over $B\backslash G$. This summing gives

$$cq_{sw} = c\,\mathrm{card}(B\backslash BswB) = c\sum_{g\in B\backslash G}\mathrm{ch}_{BswB}(g)$$

$$= c\sum_{g\in B\backslash G}(\mathrm{ch}_{BsB}*\mathrm{ch}_{BwB})(g) = \sum_{g\in B\backslash G}\sum_{h\in B\backslash G}\mathrm{ch}_{BsB}(gh^{-1})\mathrm{ch}_{BwB}(h)$$

$$= \sum\sum \mathrm{ch}_{BsB}(g)\mathrm{ch}_{BwB}(h) = q_s\,q_w$$

(the latter by replacing g by gh, interchanging order of summation).

Thus, we obtain $c = q_sq_w/q_{sw}$ and for $\ell(sw) > \ell(w)$

$$\mathrm{ch}_{BsB}*\mathrm{ch}_{BwB} = q_sq_wq_{sw}^{-1}\mathrm{ch}_{BswB}$$

This shows incidentally that the cardinality q_{sw} of $B\backslash BwB$ is finite for all $w \in W$, and therefore that the Hecke algebra really is closed under convolution.

Now we consider the other identity required of a generic algebra. Since

$$BsB\cdot BsB = B \sqcup BsB$$

we see that we need evaluate $(T_s * T_s)(g)$ only at $g = 1$ and $g = s$. In the first case, the sum defining the convolution is

$$(\mathrm{ch}_{BsB}*\mathrm{ch}_{BsB})(1) = \sum_{h\in B\backslash G}\mathrm{ch}_{BsB}(h^{-1})\mathrm{ch}_{BsB}(h) = q_s =$$

$$= (q_s-1)\cdot 0 + q_s\cdot 1 = (q_s-1)\mathrm{ch}_{BsB}(1) + q_s\mathrm{ch}_B(1)$$

In the second case,

$$(\mathrm{ch}_{BsB}*\mathrm{ch}_{BsB})(s) = \sum_{h\in B\backslash G}\mathrm{ch}_{BsB}(sh^{-1})\mathrm{ch}_{BsB}(h) =$$

$$= \mathrm{card}(B\backslash(BsB\cap BsBs))$$

Let P be the parabolic subgroup $P = B\cup BsB$. This is the stabilizer of the facet F_s. The innocent fact that P is a group allows us to compute:

$$BsB\cap BsBs = (P-B)\cap(P-B)s = (P-B)\cap(Ps-Bs) =$$

$$= (P-B)\cap(P-Bs) = P-(B\sqcup Bs)$$

Therefore, $BsB\cap BsBs$ consists of $[P:B]-2$ left B-cosets. This number is $(q_s+1)-2 = q_s-1$. Thus, altogether,

$$\mathrm{ch}_{BsB}*\mathrm{ch}_{BsB} = (q_s-1)\mathrm{ch}_{BsB} + q_s\mathrm{ch}_B$$

Therefore, already we can see that with $T_w = q_w^{-1}\mathrm{ch}_{BwB}$ we obtain a generic algebra with structure constants $a_s = (1-q_s^{-1})$ and $b_s = q_s^{-1}$. However, this is a weaker conclusion than we desire: we wish to prove further that for $\ell(sw) > \ell(w)$ we have

$$q_sq_w = q_{sw}$$

If the latter equality were true, then our earlier computation would show that, in fact,

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = \mathrm{ch}_{BswB}$$

Then taking simply $T_w = \mathrm{ch}_{BwB}$ would yield a generic algebra with structure constants $a_s = q_s - 1$ and $b_s = q_s$.

On one hand, (with $\ell(sw) > \ell(w)$) we evaluate both sides of

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = q_s q_w q_{sw}^{-1} \mathrm{ch}_{BswB}$$

at the point sw: the left hand side is

$$\sum_{h \in B\backslash G} \mathrm{ch}_{BsB}(swh^{-1})\mathrm{ch}_{BwB}(h) = \mathrm{card}(B\backslash(BsB(sw) \cap BwB)) =$$

$$= \mathrm{card}(B\backslash(BsBs \cap BwBw^{-1})) \geq \mathrm{card}(B\backslash(Bss \cap Bww^{-1})) = \mathrm{card}(B\backslash B) = 1$$

The right-hand side is simply $q_s q_w q_{sw}^{-1}$, so we have

$$q_s q_w \geq q_{sw}$$

On the other hand, invoking the theorem on generic algebras, (still with $\ell(sw) > \ell(w)$) we have

$$q_s^{-1}\mathrm{ch}_{BsB} * q_{sw}^{-1}\mathrm{ch}_{BswB} = (1 - q_s^{-1})q_{sw}^{-1}\mathrm{ch}_{BswB} + q_s^{-1}q_w^{-1}\mathrm{ch}_{BwB}$$

This gives

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BswB} = (q_s - 1)\mathrm{ch}_{BswB} + q_{sw}q_w^{-1}\mathrm{ch}_{BwB}$$

Now we evaluate both sides at w: the right side is $q_{sw}q_w^{-1}$, while the left is

$$\begin{aligned}\mathrm{card}(B\backslash(BsBw \cap BswB)) &= \mathrm{card}(B\backslash(BsB \cap BswBw^{-1})) \\ &= \mathrm{card}(B\backslash(BsB \cap BsBBwB \cdot w^{-1})) \\ &\geq \mathrm{card}(B\backslash(BsB \cap BsBww^{-1})) \\ &= \mathrm{card}(B\backslash BsB) = q_s\end{aligned}$$

by invoking the cell multiplication rules. That is, we conclude that

$$q_{sw} \geq q_s q_w$$

Combining these two computations, we have $q_{sw} = q_s q_w$ as claimed. An induction on length gives the assertion

$$q_{s_1 \ldots s_n} = q_{s_1} \ldots q_{s_n}$$

for a reduced expression $s_1 \ldots s_n \in W$. Thus, we obtain the simpler generic algebra set-up, as claimed. ♣

6.3 Generalized Iwahori-Hecke algebras

Now we consider *generalized* BN-pairs and associated convolution algebras. The necessity of considering a generalized (rather than strict) BN-pair occurs already for $GL(n)$ and the *affine* BN-pair, that is, where B is an *Iwahori subgroup*.

Let G be a group acting strongly transitively on a thick building X. Let G_o be the subgroup preserving a labeling, and *suppose that G_o still acts strongly transitively*. Let B_o be the stabilizer of a fixed chamber C *in the smaller group* G_o. We assume always that S is finite.

Fix a field k of characteristic zero. Let

$$\mathcal{H} = \mathcal{H}_k(G, B_o)$$

be the '*Hecke algebra of level* B_o' in G over the field k, that is, the collection of left and right B_o-invariant k-valued functions on G which are supported on finitely-many cosets $B_o g$ in G.

We have a *convolution product* on $\mathcal{H}$, given by

$$(f * \phi)(g) = \sum_{h \in B_o \backslash G} f(gh^{-1})\phi(h)$$

The hypothesis that ϕ is supported on finitely-many cosets $B_o x$ implies that the sum in the previous expression is *finite*. Since ϕ is left B_o-invariant and f is right B_o-invariant the summands are constant on cosets $B_o g$, so summing over $B_o \backslash G$ makes sense.

Let $\mathcal{H}_o$ be the subalgebra of functions in $\mathcal{H}$ with support inside G_o. The result of the previous section is that $\mathcal{H}_o$ is a *generic algebra*, with structure constants a_s, b_s having meaning in terms of the building, as indicated there.

Our goal in this section is to take the generic-algebra structure of $\mathcal{H}_o$ for granted, and describe the structure of $\mathcal{H}$ in terms of $\mathcal{H}_o$ and Ω.

As usual, let ch_E be the characteristic function of a set E.

Let $\mathcal{N}_o$ be the stabilizer of a chosen apartment A *in the smaller group* G_o and let $\mathcal{N}$ be the stabilizer of A *in the larger group* G. Let $T = \mathcal{N} \cap B$ and $T_o = \mathcal{N}_o \cap B_o$. From our earlier discussion of generalized BN-pairs, G_o is a normal subgroup of G, and

$$G = TG_o$$

Put

$$\Omega = T/T_o$$

Then we have a semi-direct product

$$\mathcal{N}/T_o \approx \Omega \lhd \times W$$

Thus, also $G/G_o \approx \Omega$. Define $W = \mathcal{N}_o/T_o$ as usual. For $\sigma \in \Omega$ and $w \in W$, we have

$$\sigma B_o = B_o \sigma = B_o \sigma B_o$$

$$\sigma B_o w B_o = B_o \sigma w B_o = B_o(\sigma w \sigma^{-1}) B_o \sigma$$

where we note that $\sigma w \sigma^{-1} \in W$.

Let (W, S) be the Coxeter system associated to the apartments: each apartment is isomorphic to the Coxeter complex of this pair (W, S). *We assume as in the previous section that for all $s \in S$ the cardinality*

$$q_s = \mathrm{card}(B_o s B_o / B_o) = \mathrm{card}(B_o \backslash B_o s B_o)$$

is finite. Again, $B_o s B_o / B_o$ is in bijection with the set of chambers s-adjacent to C (other than C itself), by $g \to gC$.

Let $k[\Omega]$ be the group algebra of Ω, in the sense recalled in (6.1). Since S is assumed finite and since $G/G_o \approx \Omega$, Ω is finite. The following proposition reduces study of a part of the generalized Iwahori-Hecke algebra to a much more elementary issue:

Proposition: The subalgebra $\mathcal{H}_\Omega$ of $\mathcal{H}$ consisting of functions supported on cosets of the form $B_o\sigma$ for $\sigma \in \Omega$ is isomorphic to the group algebra $k[\Omega]$, by the map

$$\mathrm{ch}_{B_o \sigma B_o} = \mathrm{ch}_{B_o \sigma} \to [\sigma] \in k[\Omega]$$

Proof: This is a nearly trivial exercise, using the properites of generalized BN-pairs recalled just above. ♣

Now define an action of Ω on $\mathcal{H}_o$ by

$$\mathrm{ch}^\sigma_{B_o w B_o} = \mathrm{ch}_{B_o(\sigma^{-1} w \sigma) B_o}$$

for $w \in W$. We introduce a 'twisted' multiplication in $k[\Omega] \otimes_k \mathcal{H}_o$ by

$$([\sigma] \otimes \phi)([\tau] \otimes \psi) = [\sigma\tau] \otimes (\phi^\tau * \psi)$$

and denote the tensor product with this multiplication by

$$k[\Omega] \otimes' \mathcal{H}_o$$

The main point here is

Theorem: The generalized Iwahori-Hecke algebra $\mathcal{H}$ is

$$\mathcal{H} \approx k[\Omega] \otimes' \mathcal{H}_o$$

with isomorphism given by the map

$$\mathrm{ch}_{B\sigma w B} \to [\sigma] \otimes \mathrm{ch}_{BwB}$$

Proof: We take for granted the structural results (6.2) on the *strict* Iwahori-Hecke algebra $\mathcal{H}_o$. The key point here is that

$$\mathrm{ch}_{B_o\sigma} * \mathrm{ch}_{B_o w B_o} = \mathrm{ch}_{B_o \sigma w B_o}$$

This is direct computation: for $g \in G$ so that the convolution does not vanish, and for $h \in G$ so that the h^{th} summand in the convolution does not vanish, we have

$$g = (gh^{-1})h \in (B_o\sigma)(B_o w B_o) = B_o \sigma w B_o$$

Thus, the convolution is *some* multiple of $\mathrm{ch}_{B_o \sigma w B_o}$. Take $g = \sigma w$ without loss of generality, to compute the convolution. The summand in the convolution is non-zero only for both $(\sigma w)h^{-1} \in B_o \sigma B_o$ and $h \in B_o w B_o$. That is, it is non-zero only for

$$h \in (B_o \sigma^{-1} B_o \sigma w) \cap (B_o w B_o) = (B_o \sigma^{-1} \sigma B_o w) \cap (B_o w B_o) =$$
$$= B_o w \cap B_o w B_o = B_o w$$

Thus, the sum over $h \in B_o \backslash G$ has non-zero summand only for $h \in B_o w$. That is, the convolution is exactly $\mathrm{ch}_{B_o \sigma w B_o}$, as claimed.

A similar computation shows that for $w \in W$ and $\sigma \in \Omega$

$$\mathrm{ch}_{B_o w B_o} * \mathrm{ch}_{B_o \sigma B_o} = \mathrm{ch}_{B_o w \sigma B_o}$$

Further, this is equal to

$$\mathrm{ch}_{B_o \sigma (\sigma^{-1} w \sigma) B_o} = \mathrm{ch}_{B_o \sigma B_o} * \mathrm{ch}_{B_o (\sigma^{-1} w \sigma) B_o}$$

Thus, it is easy to check that the multiplication in $\mathcal{H}$ is indeed the 'twisted' tensor product multiplication as defined above. ♣

Remarks: The previous theorem is to be interpreted as having successfully reduced study of *generalized* Iwahori-Hecke algebras to that of *strict* ones, mediated by the relatively *elementary* group algebra $k[\Omega]$ and its action on the strict Iwahori-Hecke algebra. But as it stands, the previous theorem only prepares for the *beginning* of such study.

Geometric Algebra

- $GL(n)$ (a prototype)
- Bilinear and hermitian forms: classical groups
- A Witt-type theorem: extending isometries
- Parabolics, unipotent radicals, Levi components

In this part we set up standard geometric algebra. This is completely independent of previous developments concerning buildings and BN-pairs, rather being preparation for three important classes of examples of *application* of the building-theory.

Note that while the terminology here is the antecedent of the Coxeter group, building, and BN-pair nomenclature above, the connections between the two require proof, which is given following each construction.

Regarding matrix notation: for a rectangular matrix R, let R_{ij} be the $(i,j)^{\text{th}}$ entry. Let R^op be the *transpose* of R, that is, $(R^op)_{ij} = R_{ji}$. If the entries of R are in a ring D and σ is an involution on D, let R^σ be the matrix with $(R^\sigma)_{ij} = R^\sigma_{ji}$.

7.1 GL(n) (a prototype)

The group $GL(n)$ is the classical group most easily studied, but already indicates interesting and important phenomena to be anticipated in other situations.

The **general linear group** $GL(n,k)$ is the group of invertible $n \times n$ matrices with entries in a field k. The **special linear group** $SL(n,k)$ is the group of (invertible) $n \times n$ matrices with entries in a field k and having determinant 1.

If we wish a less coordinate-dependent style of writing, we fix an n-dimensional k-vectorspace V and let $GL_k(V)$ be the group of k-linear automorphisms of V.

Any choice of k-basis for V gives an isomorphism $GL_k(V) \to GL(n,k)$, by taking the matrix of a linear transformation with respect to the chosen basis. Let $e_1, \ldots, e_n$ be the **standard basis** for k^n:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 \\ 1 \\ \ldots \\ 0 \end{pmatrix} \quad \ldots \quad e_n = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ 1 \end{pmatrix}$$

By this choice of (ordered) basis we obtain an isomorphism

$$GL_k(k^n) \to GL(n,k)$$

A **flag** in V is a chain

$$V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_m}$$

of subspaces, where V_i is of dimension i and

$$d_1 < \ldots < d_m$$

We say that the **type** of the flag is the sequence $(d_1, \ldots, d_m)$. In k^n the **standard flag of type** $(d_1, \ldots, d_m)$ is the flag of type $(d_1, \ldots, d_m)$ with

$$V_{d_i} = ke_1 + ke_2 + \ldots + ke_{d_i - 1} + ke_{d_i}$$

A **parabolic subgroup** $P = P_{\mathcal{F}}$ in $GL_k(V)$ is the stabilizer of a flag

$$\mathcal{F} = (V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_m})$$

That is,

$$P_{\mathcal{F}} = \{g \in GL_k(V) : gV_{d_i} = V_{d_i} \ \forall i\}$$

If $V = k^n$ and $\mathcal{F}$ is the standard flag of type $(d_1, \ldots, d_m)$, then the parabolic $P_{\mathcal{F}}$ consists of all elements admitting a block decomposition

$$\begin{pmatrix} d_1 \times d_1 & & & * \\ & (d_2 - d_1) \times (d_2 - d_1) & & \\ & & \ddots & \\ 0 & & & (d_m - d_{m-1}) \times (d_m - d_{m-1}) \end{pmatrix}$$

where (as indicated) the i^{th} diagonal entry is $(d_i - d_{i-1}) \times (d_i - d_{i-1})$, the lower entries are 0, and the entries above the diagonal blocks are arbitrary.

Each $g \in P = P_{\mathcal{F}}$ induces a natural map on the quotients $V_{d_i}/V_{d_{i-1}}$ (where we define $V_{d_o} = 0$ and $V_{d_{m+1}} = V$). Then the **unipotent radical** $R_u P$ is

$$R_u P = \{p \in P_{\mathcal{F}} : p = \text{id on all } V_{d_i}/V_{d_{i-1}} \text{ and on } V/V_{d_m}\}$$

The unipotent radical $R_u P$ is a normal subgroup of P.

In the case of the standard parabolic P of type $(d_1, \ldots, d_m)$ on k^n, the unipotent radical consists of elements which look like

$$\begin{pmatrix} 1_{d_1} & * & \ldots & & \\ & 1_{d_2} & * & \ldots & \\ & & \ddots & * & \ldots \\ & & & \ddots & * \\ 0 & & & & 1_{d_m} \end{pmatrix}$$

where 1_d denotes the identity matrix of size $d \times d$.

Choose subspaces V'_{n-d_i} of V so that V'_{n-d_i} is a complementary subspace to V_{d_i} in V and so that

$$V'_{n-d_m} \subset \ldots \subset V'_{n-d_1}$$

is a flag of **opposite type** to the flag of V_{d_i}'s. Put

$$P' = \{g \in GL_k(V) : gV'_{n-d_i} = V'_{n-d_i} \ \forall i\}$$

$$M = P \cap P'$$

Then M is called a **Levi component** or **Levi complement** in P, and $P = P_{\mathcal{F}}$ is the semidirect product

$$P = M \lhd \times R_u P$$

of M and $R_u P$, where M normalizes $R_u P$.

For the standard parabolic P in $GL(n,k)$ of type $(d_1, \ldots, d_m)$, the standard choice of the complementary spaces is

$$V'_{n-d_i} = ke_{d_i+1} + \ldots + ke_n$$

Then the standard Levi component is the group of matrices of the form

$$\begin{pmatrix} d_1 \times d_1 & & & & 0 \\ & d_2 \times d_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & d_m \times d_m \end{pmatrix}$$

where (as indicated) the i^{th} diagonal entry is $d_i \times d_i$, and all other blocks are zeros.

In the case of $GL_k(V)$, the Levi components of *minimal* parabolics are called **maximal (k-)split tori**.

The following result is a prototype for the analogous results for larger classes of groups.

Proposition:

- All parabolics of a given type are conjugate in $GL_k(V)$
- All Levi components in a parabolic subgroup P are conjugate by elements of P.
- All maximal k-split tori are conjugate in $GL_k(V)$.

Proof: This proof is a paradigm...

To prove the conjugacy of parabolics of a given type, it suffices to show that for two flags

$$V_{d_1} \subset \ldots \subset V_{d_m}$$
$$V'_{d_1} \subset \ldots \subset V'_{d_m}$$

of the same type there is $g \in GL_k(V)$ so that $gV_{d_i} = V'_{d_i}$ for all i. *Choose* two bases $\{v_i\}, \{v'_i\}$ for V, so that

$$V_{d_i} = kv_1 + \ldots + kv_{d_i}$$
$$V'_{d_i} = kv'_1 + \ldots + kv'_{d_i}$$

Then define g by $gv_i = v'_i$. This proves the conjugacy of parabolics of a given type.

To prove conjugacy of Levi components within a given parabolic P, let

$$V_{d_1} \subset \ldots \subset V_{d_m}$$

be the flag of which P is the stabilizer, and let $V^1_{n-d_i}, V^2_{n-d_i}$ (with $1 \le i \le m$) be two choices of families of complementary subspaces defining (in our present terms) Levi components. It suffices to find $p \in P$ so that $pV^1_{n-d_i} = V^2_{n-d_i}$ for all indices i.

For $\ell = 1, 2$, define $W^\ell_1, \ldots, W^\ell_{m+1}$ to be, respectively,

$$V_{d_1},\ V_{d_2} \cap V^\ell_{n-d_1},\ V_{d_3} \cap V^\ell_{n-d_2}, \ldots,\ V_{d_m} \cap V^\ell_{n-d_{m-1}},\ V^\ell_{n-d_m}$$

For $\ell = 1, 2$ we have $V = \oplus W^\ell_i$. By hypothesis, $\dim_k W^1_j = \dim_k W^2_j$ for all j; therefore, there are many elements $g \in GL_k(V)$ so that $gW^1_j = W^2_j$ for all j. For any such g, certainly $g \in P$, and since V^ℓ_j is a sum of W^ℓ_i's, certainly $pV^1_{n-d_i} = V^2_{n-d_i}$ for all i.

Given two maximal split tori T_1, T_2, choose minimal parabolics P_i containing T_i. By the first part of the proposition, there is $h \in GL_k(V)$ so that $hP_1h^{-1} = P_2$. Then hT_1h^{-1} is another Levi component (maximal split torus) inside P_2, so by the second assertion of this proposition there is $p \in P_2$ so that $p(hT_1h^{-1})p^{-1} = T_2$. This gives the third assertion of the proposition. ♣

Now we generalize the previous in a mostly straightforward way: replace the *field* k of the previous section by a *division ring* D. We repeat the coordinate-free version of the previous discussion; the matrix pictures are identical to those just above.

We define a finite-dimensional vectorspace V over a division ring D to be a finitely-generated free module over D. The notion of *dimension* makes sense, being defined as *rank* of a free module. Elementary results about linear independence and bases are the same as over fields.

The loss of commutativity of D becomes relevant when considering D-linear endomorphisms. If D is not commutative, then the ring $\mathrm{End}_D(V)$ of D-linear endomorphisms of V does *not* naturally contain D. Thus, a choice of D-basis for an n-dimensional D vectorspace V gives an isomorphism

$$\mathrm{End}_D(V) \to \{n \times n \text{ matrices with entries in } D^{\mathrm{opp}}\}$$

where D^{opp} is the *opposite ring* to D. That is, D^{opp} is the same additive group as D, but with multiplication $*$ given by

$$x * y = yx$$

where yx is the multiplication in D.

(Sometimes this (harmless) complication is avoided by declaring V to be a *'right'* D-module, but the definition of 'right' module really is that of module over the opposite ring D^{opp} anyway.)

The **general linear group** $GL(n, D)$ over D is the group of invertible $n \times n$ matrices with entries in D. The coordinate-free version of the general linear group is $GL_D(V)$, the group of D-linear automorphisms of V. Choice of D-basis for V gives an isomorphism

$$GL_D(V) \to GL(n, D^{\mathrm{opp}})$$

Definitions regarding *flags* and *parabolics* are identical to those in the case that D was a field:

A **flag** in V is a chain

$$V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_m}$$

of subspaces, where V_i is of D-dimension i and

$$d_1 < \ldots < d_m$$

The **type** of the flag is the sequence $(d_1, \ldots, d_m)$.

A **parabolic subgroup** $P = P_{\mathcal{F}}$ in $GL_D(V)$ is the stabilizer of a flag

$$\mathcal{F} = (V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_m})$$

That is,

$$P_{\mathcal{F}} = \{g \in GL_D(V) : gV_{d_i} = V_{d_i} \ \forall i\}$$

Each $g \in P = P_{\mathcal{F}}$ induces a natural map on the quotients $V_{d_i}/V_{d_{i-1}}$ (where we define $V_{d_o} = 0$ and $V_{d_{m+1}} = V$). The **unipotent radical** R_uP is

$$R_uP = \{p \in P_{\mathcal{F}} : p = \text{id on } V_{d_i}/V_{d_{i-1}} \ \forall i\}$$

The unipotent radical R_uP is a normal subgroup of P.

Choose subspaces V'_{n-d_i} of V so that V'_{n-d_i} is a complementary subspace to V_{d_i} in V. Then

$$V'_{n-d_m} \subset \ldots \subset V'_{n-d_1}$$

is a flag of **opposite type** to the flag of V_{d_i}'s. Put

$$P' = \{g \in GL_D(V) : gV'_{n-d_i} = V'_{n-d_i} \ \forall i\}$$

$$M = P \cap P'$$

Then M is called a **Levi component** or **Levi complement** in P, and $P = P_{\mathcal{F}}$ is the semidirect product

$$P = M \lhd \times R_uP$$

of M and R_uP, where M normalizes R_uP.

Proposition:

- All parabolics of a given type are conjugate in $GL_D(V)$
- All Levi components in a parabolic subgroup P are conjugate by elements of P.

The proofs of these assertions are identical to those for $GL(n, k)$. ♣

7.2 Bilinear and hermitian forms, classical groups

In this section we introduce the classical groups defined as isometry or similitude groups of 'forms' on vectorspaces. We define orthogonal groups and symplectic groups first, then the unitary groups, and then more general groups including what are sometimes denoted as O^* and Sp^*. (This family of descriptions could be simplified, at the cost of obscuring the simpler members.)

Let k be a field not of characteristic 2, and let V be a finite-dimensional k-vectorspace. A (k-)**bilinear form** on V is a k-valued function on $V \times V$ so that, for all $x, y \in k$ and $v, v_1, v_2 \in V$

$$\langle v_1 + v_2, v\rangle = \langle v_1, v\rangle + \langle v_2, v\rangle$$

$$\langle v, v_1 + v_2\rangle = \langle v, v_1\rangle + \langle v, v_2\rangle$$

$$\langle xv, yv_1\rangle = xy\langle v, v_1\rangle$$

If always

$$\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$$

then the bilinear form is **symmetric**. The function

$$Q[v] = \langle v, v\rangle$$

is the **associated quadratic form**, from which $\langle,\rangle$ can be recovered by

$$4\langle v_1, v_2\rangle = Q[v_1 + v_2] - Q[v_1 - v_2]$$

The associated **orthogonal group** is the **isometry group** of Q or $\langle,\rangle$, which is defined as

$$O(Q) = O(\langle,\rangle) = \{g \in GL_k(V) : \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

The associated **similitude group** is defined as

$$GO(Q) = GO(\langle,\rangle) = \{g \in GL_k(V) : \exists\, \nu(g) \in k^\times \text{ so that } \langle gv_1, gv_2\rangle = \nu(g)\,\langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

If always

$$\langle v_1, v_2\rangle = -\langle v_2, v_1\rangle$$

then the bilinear form is **alternating** or **symplectic** or **skew-symmetric**. The associated **symplectic group** is the **isometry group** of $\langle,\rangle$, which is defined as

$$Sp(\langle,\rangle) = \{g \in GL_k(V) : \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

The associated *similitude group* is defined as

$$GSp(\langle,\rangle) = \{g \in GL_k(V) : \exists\, \nu(g) \in k^\times \text{ so that } \langle gv_1, gv_2\rangle = \nu(g)\,\langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

Let K be a quadratic field extension of k with non-trivial k-linear automorphism σ. A k-bilinear form $\langle , \rangle$ on a finite-dimensional K-vectorspace V is **hermitian** (with implicit reference to σ) if

$$\langle xv_1, yv_2\rangle = xy^\sigma \langle v_1, v_2\rangle$$

for all $x, y \in K$ and for all $v_1, v_2 \in V$. The associated **unitary group** is the **isometry group** of $\langle , \rangle$, which is defined as

$$U(\langle , \rangle) = \{g \in GL_K(V) : \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

The associated *similitude group* is defined as

$$GU(\langle , \rangle) = \{g \in GL_K(V) : \exists\, \nu(g) \in k^\times \text{ so that } \langle gv_1, gv_2\rangle =$$

$$\nu(g)\, \langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

The previous groups can all be treated simultaneously, while also including more general ones, as follows.

Let D be a **division algebra** with **involution** σ. That is, $\sigma : D \to D$ has properties

$$(\alpha)^{\sigma\sigma} = \alpha \quad\text{and}\quad (\alpha + \beta)^\sigma = \alpha^\sigma + \beta^\sigma \quad\text{and}\quad (\alpha\beta)^\sigma = \beta^\sigma\, \alpha^\sigma$$

for all $\alpha, \beta \in D$. Let Z be the center of D. We require that D is finite-dimensional over Z. Certainly σ stabilizes Z. If σ is *trivial* on Z then say that σ is an **involution of first kind**; if σ is *non-trivial* on Z then say that σ is an **involution of second kind**. In either case, we suppose that

$$k = \{x \in Z : x^\sigma = x\}$$

Let V be a finite-dimensional vectorspace over D, and fix $\epsilon = \pm 1$. Let

$$\langle , \rangle : V \times V \to D$$

be a D-valued k-bilinear 'form' on V so that

$$\langle v_2, v_1\rangle = \epsilon \langle v_1, v_2\rangle^\sigma$$

$$\langle \alpha v_1, \beta v_2\rangle = \alpha \langle v_1, v_2\rangle \beta^\sigma$$

for all $\alpha, \beta \in D$ and $v_1, v_2 \in V$. This is an ϵ-**hermitian form** on V. For want of better terminology, we call V (equipped with $\langle , \rangle$) a (D, σ, ϵ)**-space**.

Let V_i be (D, σ, ϵ)-spaces with forms $\langle , \rangle_i$ (for $i = 1, 2$). A D-linear map $\phi : V_1 \to V_2$ is an **isometry** if, for all $u, v \in V_1$,

$$\langle \phi u, \phi v\rangle_2 = \langle u, v\rangle$$

The map ϕ is a **similitude** if there is $\nu \in k$ so that, for all $u, v \in V_1$,

$$\langle \phi u, \phi v\rangle_2 = \nu \langle u, v\rangle$$

Write $\phi : V_1 \cong V_2$ if ϕ is an isometry.

The associated **isometry group** of $\langle , \rangle$ is defined as

$$\{g \in GL_D(V) : \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \ \forall v_1, v_2 \in V\}$$

The associated *similitude group* is defined as

$$\{g \in GL_D(V) : \exists\, \nu(g) \in k^\times \text{ so that } \langle gv_1, gv_2\rangle = \nu(g)\,\langle v_1, v_2\rangle\ \forall v_1, v_2 \in V\}$$

A D-subspace U of a (D,σ,ϵ)-space V has **orthogonal complement**

$$U^\perp = \{u' \in V : \langle u', u\rangle = 0\ \forall u \in U\}$$

Note that there is in general no assurance that $U \cap U^\perp = 0$. The **kernel** of the whole space V is $V^\perp$. The form is **non-degenerate** if $V^\perp = 0$. *Often we will suppress reference to the form and say merely that the space V itself is non-degenerate. Such abuse of language is typical in this subject.*

If V_1, V_2 are two (D,σ,ϵ)-spaces with respective forms $\langle,\rangle_1, \langle,\rangle_2$, then the direct sum $V_1 \oplus V_2$ of D-vectorspaces is a (D,σ,ϵ)-space with form

$$\langle v_1 + v_2, v_1' + v_2'\rangle = \langle v_1, v_1'\rangle_1 + \langle v_2, v_2'\rangle_2$$

We call this an **orthogonal sum**. Generally, two subspaces V_1, V_2 of a (D,σ,ϵ)-space are **orthogonal** if

$$V_1 \subset V_2^\perp$$

or equivalently, if $V_2 \subset V_1^\perp$.

If $\langle v, v\rangle = 0$ for $v \in V$, then v is an **isotropic vector** . If $\langle v, v'\rangle = 0$ for all $v, v' \in U$ for a subspace U of V, then U is a **(totally) isotropic subspace**. If *no* non-zero vector in U is isotropic, then U is **anisotropic**.

Proposition: Let V be a non-degenerate (D,σ,ϵ)-space with subspace U. Then U is non-degenerate if and only if $V = U \oplus U^\perp$, if and only if $U^\perp$ is non-degenerate.

Proof: We map $\Lambda : V \to \mathrm{Hom}_D(U, D)$ by $v \to \lambda_v$ where

$$\lambda_v(u) = \langle u, v\rangle$$

The non-degeneracy of V assures that Λ is onto. The kernel is visibly $U^\perp$. Then, by linear algebra,

$$\dim_D\ U^\perp + \dim_{D^{\mathrm{opp}}}\ \Lambda(U) = \dim_D\ V$$

Thus, since the dimension of $\Lambda(V)$ is the same as the dimension of U, by dimension-counting, $U \cap U^\perp = 0$ if and only if $U + U^\perp$ is a *direct* (and hence orthogonal) sum.

Since $U \subset U^{\perp\perp}$, U degenerate implies that $U \cap U^\perp$ is non-zero. Then $U^\perp \cap U^{\perp\perp}$ is non-zero, since it contains $U \cap U^\perp$, so $U^\perp$ is degenerate. On the other hand, U non-degenerate implies that $U + U^\perp$ is a direct sum, so $\dim\ U = \dim\ V - \dim U^\perp$. Since $\dim\ U^{\perp\perp} = \dim\ V - \dim U^\perp$ by the non-degeneracy of V, we have $U^{\perp\perp} = U$, so $U^{\perp\perp} + U^\perp$ is a direct sum, and $U^\perp$ is non-degenerate. ♣

A D-basis $e_1, \ldots, e_n$ for a (D,σ,ϵ)-space V is an **orthogonal basis** if $\langle e_i, e_j\rangle = 0$ for $i \neq j$.

Proposition: Let V be a (D, σ, ϵ)-space. Exclude the case that $\epsilon = -1, D = k$, and σ is trivial. If $\langle , \rangle$ is not identically zero then there is $v \in V$ with $\langle v, v \rangle \neq 0$. If V is non-degenerate then it has an orthogonal basis.

Proof: Suppose that $\langle v, v \rangle = 0$ for all $v \in V$. Then

$$0 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \epsilon \langle x, y \rangle^\sigma + \langle y, y \rangle = \langle x, y \rangle + \epsilon \langle x, y \rangle^\sigma$$

If $\epsilon = 1$ and $\langle , \rangle$ is not identically 0, then there are x, y so that $\langle x, y \rangle = 1$. Then we have

$$0 = \langle x, y \rangle + \epsilon \langle x, y \rangle^\sigma = 1 + 1$$

contradiction.

Suppose that $\epsilon = -1$ and σ is not trivial on D. Then there is $\alpha \in D$ so that $\alpha^\sigma \neq \alpha$, and with $\omega = \alpha - \alpha^\sigma$, $\omega^\sigma = -\omega$. If $\langle , \rangle$ is not identically 0, then there are x, y so that $\langle x, y \rangle = 1$. Then we have

$$0 = \langle \omega x, y \rangle + \epsilon \langle \omega x, y \rangle^\sigma = \omega \langle x, y \rangle - \langle x, y \rangle^\sigma \omega^\sigma =$$

$$= \omega - \epsilon\omega = 2\omega$$

contradiction.

To construct an orthogonal basis, do induction on dimension. If the dimension of a non-degenerate V is 1, then any non-zero element is an orthogonal basis. Generally, by the previous discussion, we can find $v \in V$ so that $\langle v, v \rangle \neq 0$. Then $Dv^\perp$ is non-degenerate and V is the orthogonal direct sum of Dv and $Dv^\perp$, by the previous proposition. ♣

Suppose that V is two-dimensional, with an ordered basis x, y so that

$$\langle x, x \rangle = \langle y, y \rangle = 0 \quad \text{and} \quad \langle x, y \rangle = 1$$

Then V is a **hyperbolic plane** and x, y is a **hyperbolic pair** in V. A (D, σ, ϵ)-space is **hyperbolic** if it is an orthogonal sum of hyperbolic planes.

Proposition: Let V and W be two hyperbolic spaces of the same dimension (with the same data D, σ, ϵ). Then there is an isometry $f : V \to W$. That is, dimension is the only invariant of hyperbolic spaces.

Proof: Match up hyperbolic pairs. ♣

Proposition: Take V non-degenerate with $\epsilon = -1, D = k$, and σ trivial. Then V is *hyperbolic*, that is, is an orthogonal sum of hyperbolic planes.

Proof: Since σ is trivial, $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in D$, so D is a field. Since

$$\langle x, x \rangle = -\langle x, x \rangle$$

and the characteristic is not 2, every vector is isotropic. Fix $x \in V$ non-zero, and take $y \in V$ so that $\langle x, y \rangle \neq 0$. Then by changing y by an element of D we can make $\langle x, y \rangle = 1$, that is, a hyperbolic pair. Then $Dx + Dy$ and $(Dx + Dy)^\perp$ are non-degenerate, and we do induction on dimension. ♣

Proposition: Let V be a non-degenerate space and $-V$ the same space with the *negative* of the form on V. Then the orthogonal sum

$$W = V \oplus -V$$

is *hyperbolic.*

Proof: In the case of (non-degenerate) alternating spaces (with $D = k, \epsilon = -1, \sigma$ trivial), V itself is already hyperbolic, and then $-V$ is visibly so. On the other hand, for a (non-degenerate) non-alternating space V, we can find an orthogonal basis $\{e_i\}$ (for both V and $-V$). Then we claim that in $V \oplus -V$ the subspaces

$$H_i = De_i \oplus De_i$$

are hyperbolic planes, for all indices i. (This would prove the proposition). Since the characteristic is not 2, we can consider the vectors

$$x_i = \frac{1}{2} e_i \oplus e_i \qquad y_i = \langle e_i, e_i \rangle^{-1} e_i \oplus -e_i$$

which are linearly independent (since $1 \neq -1$). They are both isotropic, by design. And the constants are such that for the form $\langle , \rangle$ on $V \oplus -V$ we have $\langle x_i, y_i \rangle = 1$. ♣

Proposition: Let V be non-degenerate, and W a subspace. Let W_o be the kernel of W. Then there is a non-degenerate subspace W_1 of W so that $W_o + W_1 = W$ is a direct sum. Further, for any basis $x_1, \ldots, x_n$ for W_o, and for any such W_1, there is a set $\{y_i\} \subset W_1^\perp$ so that the subspaces $Dx_i + Dy_i$ are mutually orthogonal hyperbolic planes. In particular,

$$W + \sum_i Dy_i = W \oplus (\oplus_i Dy_i)$$

is non-degenerate and $W_o + \sum_i Dy_i$ is a hyperbolic space.

Proof: The form $\langle , \rangle$ induces a non-degenerate form of the same 'type' on the quotient W/W_o. It is easy to see that this quotient is non-degenerate. Let W_1 be any vectorspace complement to W_o in W. Then the non-degeneracy of W/W_o implies that of W_1.

Since $U = W_1 + Dx_2 + \ldots + Dx_n$ is a proper subspace of W (noting that x_1 is missing), and since V is non-degenerate, $W^\perp$ is a proper subspace of $U^\perp$. That is, there is a non-zero element y in $U^\perp$ but not in $W^\perp$. Then $\langle x_1, y \rangle \neq 0$. Adjusting y by an element of D allows us to make $\langle x_1, y \rangle = 1$. Since $\langle y, y \rangle = \epsilon \langle y, y \rangle \sigma$, $y_1 = y - \frac{1}{2} \langle y, y \rangle x_1$ is the desired element to make a hyperbolic pair x_1, y_1.

Now $Y = (Dy_1)^\perp \cap W_o$ is of codimension 1 in W_o, and is the kernel of $W + Dy_1$. Thus, induction on the dimension of W_o gives the proposition. ♣

Corollary: Let V be a non-degenerate space. Then there is a hyperbolic subspace H of V and an anisotropic subspace A of V so that V is the orthogonal direct sum $V = H \oplus A$.

Proof: This is by induction on the dimension of V. If V is anisotropic, we are done. If not, let v be a non-zero isotropic vector, and by the previous proposition find another vector w so that v, w is a hyperbolic pair. Then $(Dv + Dw)^\perp$ is non-degenerate and of smaller dimension than V. ♣

7.3 A Witt-type theorem: extending isometries

Here we give a result including the traditional Witt theorem on extensions of isometries in non-degenerate 'formed' spaces. The proof here is somewhat more 'element-free' than the traditional proof. This result implies, as a special case, that all parabolic subgroups 'of the same type' in isometry (and similitude) groups are *conjugate*.

Still we exclude characteristic 2, and keep the other notation and hypothesis of previous sections.

For a (D, σ, ϵ)-space V with form $\langle , \rangle$, let $-V$ denote the (D, σ, ϵ)-space which is the same D-vectorspace but with form $-\langle , \rangle$. Let V_o denote the kernel of a (D, σ, ϵ)-space V.

Theorem:

- Let U, W be subspaces of a non-degenerate space V. Every isometry $\phi : U \to W$ extends to an isometry $\Phi : V \to V$. (That is, Φ restricted to U is ϕ).
- If U, V, W are spaces so that $U \oplus V \cong U \oplus W$, then $V \cong W$.

Proof: The main technical device in the proof is consideration of a certain configuration which occurs elsewhere as well. We introduce this first, and then proceed with the proof.

Suppose that $V = X \oplus Y$ with X, Y non-degenerate proper subspaces of V, and with V hyperbolic. Let W be a maximal totally isotropic subspace of V. From the previous section's results it follows easily that $\dim_D W = \frac{1}{2} \dim_D V$. Let A, B be the images of W under projection to X, Y, respectively. Since both X, Y are proper subspaces of V, a maximal isotropic subspace of X (or of Y) has *strictly* smaller dimension than W, so neither of A, B is 0. Since W is maximal isotropic and V is hyperbolic, we have $W = W^\perp$, and thus the kernel of A is

$$A_o = \{x \in A : \langle x, w \rangle = 0 \; \forall w \in W\} = A \cap W^\perp = A \cap W = X \cap W$$

Similarly, the kernel of B is

$$B_o = Y \cap W$$

Define $\psi : A \to B/B_o$ by $\psi x = y + B_o$ where $x + y \in W$. Then for $x, x' \in A$,

$$\langle \psi x, \psi x' \rangle = \langle \psi x, x' + \psi x' \rangle =$$

$$= \langle \psi x + x - x, x' + \psi x' \rangle = \langle -x, x' + \psi x' \rangle = -\langle x, x' \rangle$$

That is, ψ induces an isometry (also denoted by ψ)

$$A/A_o \overset{\cong}{\to} -B/B_o$$

Note that both A/A_o and B/B_o are non-degenerate.

From the discussion of the previous section, there are totally isotropic subspaces A' of X and B' of Y so that both $A_o \oplus A'$ and $B_o \oplus B'$ are hyperbolic, and so that $A \cap A' = B \cap B' = 0$.

Lemma:

$$X = A \oplus A' \quad \text{and} \quad Y = B \oplus B'$$

Proof: To prove this, let

$$2N = \dim\ V = 2\dim\ W = 2(\dim\ A_o + \dim B) = 2(\dim\ A + \dim\ B_o$$

$$m = \dim\ A_o \quad n = \dim\ B_o \quad r = \dim\ A/A_o = \dim\ B/B_o$$

Then we have

$$2N = \dim\ X + \dim\ Y \geq (\dim A + \dim A') + (\dim B + \dim B') =$$

$$= (r + 2m) + (r + 2n) = 2(m + r + n) = 2(\dim A_o + \dim B) = 2N$$

Therefore, equality must hold, proving the claim. ♣

Lemma: In the above situation, suppose that X is anisotropic, and that Y is hyperbolic. Then $X = 0$.

Proof: The projection of W to B must be *injective*, since the kernel of this projection is $A_o = X \cap W = 0$, using the anisotropy of X. Thus, in the notation of the previous lemma, $A' = 0$. Since $X = A \oplus A'$ by the previous lemma, we see that in the present situation $X = A$.

Further, we can choose B' to lie inside $\psi X^\perp$. Then the previous lemma shows that we have a direct sum decomposition

$$Y \cong -X \oplus (B_o \oplus B')$$

where now $B_o \oplus B'$ is hyperbolic.

Let V be the hyperbolic space of least dimension so that there is Y so that $X \oplus Y \cong V$ with Y also hyperbolic. We have

$$X \oplus (-(B_o \oplus B') \cong -Y$$

with $-(B_o \oplus B')$ hyperbolic. If $X \neq 0$, this contradicts the minimality of V, since $X \neq 0$ implies $\dim Y < \dim V$. Thus, if $X \neq 0$ then there are no such Y, V. ♣

Lemma: If

$$X \oplus \text{ hyperbolic } \cong \text{ hyperbolic}$$

then X itself is hyperbolic.

Proof: Let $X = X^+ \oplus H$ with H hyperbolic and X^+ anisotropic. If $X \oplus Y \cong V$ with both Y, V hyperbolic, then we have

$$V \cong X^+ \oplus (H \oplus Y)$$

so by the previous lemma $X^+ = 0$. ♣

Lemma: If

$$U \oplus X \cong U \oplus Y$$

with U, X, Y all non-degenerate, then $X \oplus -Y$ is hyperbolic.

Proof: Certainly $H = U \oplus -U$ is hyperbolic, and

$$(X \oplus -Y) \oplus H \cong (X \oplus -Y) \oplus (U \oplus -U) \cong (U \oplus X) \oplus -(U \oplus Y) \cong$$

$$\cong (U \oplus Y) \oplus -(U \oplus Y) \text{ hyperbolic}$$

by invoking the hypothesis $U \oplus X \cong U \oplus Y$. (Always $V \oplus -V$ is hyperbolic for any non-degenerate V). Thus, by the previous lemma, we have the conclusion. ♣

In the situation of the last lemma, we write $X = X^+ \oplus H_1$ and $Y = Y^+ \oplus H_2$ with H_i hyperbolic and X^+, Y^+ anisotropic. Then since $X \oplus -Y$ is hyperbolic it follows from the lemma above that $X^+ \oplus -Y^+$ is hyperbolic. Taking a direct sum of both sides with Y^+ gives

$$X^+ \oplus (-Y^+ \oplus Y^+) \cong Y^+ \oplus \text{ hyperbolic}$$

Now $-Y^+ \oplus Y^+$ is itself hyperbolic (for *any* non-degenerate space), so by the lemma above we have

$$X^+ \cong Y^+ \oplus \text{ hyperbolic}$$

Symmetrically,

$$Y^+ \cong X^+ \oplus \text{ hyperbolic}$$

Putting the latter two assertions together, we conclude that $X^+ \cong Y^+$. Then the hypothesis $U \oplus X \cong U \oplus Y$ assures that the dimensions of H_1, H_2 are the same, so they are isometric, being hyperbolic.

This proves the second assertion of the theorem.

We saw in the previous section that U can be orthogonally decomposed as $U \cong U^+ \oplus U_o$ where $U^+ \cong U/U_o$ is non-degenerate. As described earlier, for a basis $x_1, \ldots, x_n$ of U_o, we can choose $y_1, \ldots, y_n$ in $(U^+)^\perp$ so that each $Dx_i + Dy_i$ is a hyperbolic plane, and so that

$$\tilde{U} = U + (Dy_1 + \ldots + Dy_n)$$

is *non-degenerate*. Then $W^+ = \phi U^+$ is non-degenerate in W and is a complement to the kernel $W_o = \phi U_o$ of W. For the basis $\{\phi x_i\}$ of W_o, choose $z_1, \ldots, z_n$ in $(W^+)^\perp$ so that all the $D(\phi x_i) + Dz_i$ are hyperbolic planes. Then extend ϕ to an isometry

$$\Phi : U + (Dy_1 + \ldots + Dy_n) \to W + (Dz_1 + \ldots + Dz_n)$$

by defining $\Phi y_i = z_i$. It is easy to verify that this really is an isometry. By design, we have extended ϕ to an isometry on the somewhat larger *non-degenerate* space $\tilde{U}$, thereby reducing the first assertion of the theorem to the case that U (and, hence, W) are non-degenerate.

Then, using the non-degeneracy of U, W and the hypothesis $\phi : U \cong W$, the isometry

$$U \oplus U^\perp \cong V \cong W \oplus W^\perp$$

implies that there is an isometry $\phi' : U^\perp \cong W^\perp$, by the second assertion of the theorem (which is already proven). Then define Φ on V by

$$\Phi(u \oplus u') = \phi(u) + \phi'(u')$$

for $u \in U$ and $u' \in U \perp$. ♣

7.4 Parabolics, unipotent radicals, Levi components

Let D, ϵ, σ as above be fixed, and let V be a *non-degenerate* 'formed space' with this D, ϵ, σ. It is important that the space be non-degenerate. Let G be the *isometry group* of V, as defined earlier. The following discussion also applies, with minor modifications, to the similitude group and other related groups.

First we give the coordinate-independent definitions, and then in coordinates describe the *standard* parabolics, unipotent radicals, and Levi components.

An **isotropic flag** $\mathcal{F}$ in V is a chain

$$V_1 \subset \ldots \subset V_m$$

of *totally isotropic* subspaces V_i of V. The **type** of the flag is the ordered m-tuple of D-dimensions

$$(\dim_D V_1, \ldots, \dim_D V_m)$$

The **parabolic subgroup** $P = P_{\mathcal{F}}$ associated to an isotropic flag $\mathcal{F}$ is the *stabilizer* of the flag, that is,

$$P_{\mathcal{F}} = \{g \in G : gV_i = V_i \ \ \forall i\}$$

The **type** of the parabolic is defined to be the type of the isotropic flag.

Proposition: Any two parabolic subgroups of the same 'type' are conjugate by an element of G.

Proof: Let P and P' be the stabilizers of two isotropic flags

$$V_1 \subset \ldots \subset V_m$$
$$V'_1 \subset \ldots \subset V'_m$$

where

$$\dim V_i = \dim V'_i$$

Invoking Witt's theorem, there is h_m in the isometry group of V so that $h_m V'_m = V_m$. Since the form restricted to V_m is *zero*, there certainly is h^o_{m-1} in $\mathrm{Aut}_D V_m$ so that $h^o_{m-1} h_m V'_{m-1} = V_{m-1}$. By Witt's theorem this h^o_{m-1} extends to an isometry h_{m-1} of all of V. An induction completes the proof.

♣

Note that elements of P give well-defined maps on the quotients V_i/V_{i-1}. And, elements of P give well-defined maps on $V_{i-1}^\perp/V_i^\perp$, since

$$V_1 \subset \ldots \subset V_m \subset V_m^\perp \subset V_{m-1}^\perp \subset \ldots \subset V_1^\perp$$

Further, the form $\langle,\rangle$ on V gives a natural identification of $V_{i-1}^\perp/V_i^\perp$ with the *dual space* of V_i/V_{i-1}, by

$$\lambda_w(v + V_{i-1}) = \langle v, w\rangle$$

This duality respects the action of the isometry group.

The **unipotent radical** $R_u P$ of a parabolic $P = P_{\mathcal{F}}$ is defined to be the subgroup of G consisting of elements $p \in P$ so that the maps induced by p on all quotients V_i/V_{i-1} and on $V_m^\perp/V_m^\perp$ are *trivial*. Note that this *implies* that the natural actions on the quotients $V_{i-1}^\perp/V_i^\perp$ are also trivial, since these are dual spaces to the quotients V_i/V_{i-1} and this duality respects the G-action.

It is easy to see that the unipotent radical $R_u P$ of P is a *normal* subgroup of P.

Fix an isotropic flag

$$\mathcal{F} = (V_1 \subset \ldots \subset V_m)$$

Let

$$\mathcal{F}' = (V'_1 \subset \ldots \subset V'_m)$$

be another isotropic flag so that dim $V_i =$ dim V'_i and for each i

$$V_i + V'_i = V_i \oplus V'_i = \text{ a non-degenerate (hyperbolic) space}$$

The **Levi component** or **Levi complement** of the parabolic $P_{\mathcal{F}}$ corresponding to this choice is

$$M = \{p \in P : pV'_i = V'_i \ \forall i\} = P_{\mathcal{F}} \cap P_{\mathcal{F}'}$$

Note that this implies that $m \in M$ stabilizes each $V_i \oplus V'_i$ and stabilizes each $(V_i \oplus V'_i)^\perp$.

It is not hard to check that *a parabolic subgroup is the semi-direct product of its unipotent radical and any Levi component.*

Now we claim that *Levi components of parabolics of isometry groups are products of 'classical groups'.* That is, we are claiming that these Levi components are products of GL-type groups and of isometry groups.

More specifically, with two isotropic flags

$$\mathcal{F} = (V_1 \subset \ldots \subset V_m)$$

$$\mathcal{G} = (W_1 \subset \ldots \subset W_m)$$

related as above, we claim that the associated Levi component M is isomorphic to

$$H = GL_{d_1}(D^{\mathrm{opp}}) \times GL_{d_2-d_1}(D^{\mathrm{opp}}) \times \ldots$$
$$\ldots \times GL_{d_m-d_{m-1}}(D^{\mathrm{opp}}) \times \mathrm{Iso}((V_m + W_m)^\perp)$$

where $\mathrm{Iso}((V_m + W_m)^\perp)$ is the isometry group of $(V_m + W_m)^\perp$ and where $d_i = \dim_D V_i$.

Let

$$x_1 \in V_1, \;\; x_2 \in V_2 \cap W_1^\perp, \;\; x_3 \in V_3 \cap W_2^\perp, \ldots, \; x_m \in V_m \cap W_{m-1}^\perp$$
$$x_+ \in V_m^\perp \cap W_m^\perp$$
$$y_1 \in W_1, \;\; y_2 \in W_2 \cap V_1^\perp, \ldots, \; y_m \in W_m \cap V_{m-1}^\perp$$

An element of the associated Levi component can be decomposed into corresponding factors as

$$g_1 \times \ldots \times g_m \times g_+ \times g_1' \times \ldots g_m'$$

The requirement that this be an isometry is that

$$\langle g_i x_i, g_i' y_i \rangle = \langle x_i, y_i \rangle$$
$$\langle g_+ x_+, g_+ x_+ \rangle = \langle x_+, x_+ \rangle$$

since all other pairs of summands are pairwise orthogonal. That is, g_+ is an isometry as indicated, and g_i' is completely determined by g_i (as a kind of 'adjoint'), and g_i itself may be arbitrary in $GL_D(V_i \cap W_{i-1}^\perp)$. The D-dimension of $V_i \cap W_{i-1}^\perp$ is $d_i - d_{i-1}$, so this completes the verification of the claim.

8. Examples in Coordinates

- Symplectic groups in coordinates
- Orthogonal groups O(n,n) in coordinates
- Orthogonal groups O(p,q) in coordinates
- Unitary groups in coordinates

Having set up a sufficient amount of 'geometric algebra', we now use coordinates to describe the **standard** versions of some of the classical isometry and similitude groups, enough to suggest what can be done in all cases. Although in hindsight these matrix computations are of limited use, there seems to be considerable psychological comfort in seeing them, and operating at this level seems an unavoidable step in development of technique and viewpoint.

Again, there will be substantial redundancy in the sort of observations we make, with the purpose of making the phenomena unmistakable.

Regarding matrix notation: for a rectangular matrix R, let R_{ij} be the $(i,j)^{\text{th}}$ entry. Let $R^\top$ be the *transpose* of R, that is, $(R^\top)_{ij} = R_{ji}$. If the entries of R are in a ring D and σ is an involution on D, let R^σ be the matrix with $(R^\sigma)_{ij} = R^\sigma_{ji}$.

8.1 Symplectic groups in coordinates

Among the classical groups, beyond the prototypical GL_n, symplectic groups $Sp(n)$ are quite 'popular'. We treat the symplectic *similitude* groups $GSp(n)$ briefly at the end of this section.

We take $V = k^{2n}$, viewed as *column vectors*, and let

$$J = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

where 1_n and 0_n are the $n \times n$ identity and zero matrix, respectively. For $u, v \in V$, put

$$\langle u, v\rangle = v^\top J v$$

This is a non-degenerate alternating form on V. The **standard symplectic group** is

$$Sp_n = Sp_n(k) = \text{ isometry group of } \langle , \rangle \ =$$

$$= \{g \in GL_k(V) : \langle gu, gv\rangle = \langle u, v\rangle \ \ \forall u, v \in V\} =$$

$$= \{g \in GL_{2n}(k) : g^\top J g = J\}$$

(It is a small exercise in linear algebra to check that the last condition is equivalent to the others). Some authors write $Sp_{2n} = Sp_{2n}(k)$ for this group.

Now we use $n \times n$ blocks in matrices. Then, upon multiplying out the condition $g^\top J g = J$, the condition for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be in Sp_n is

$$c^\top a - a^\top c = 0 \quad d^\top b - b^\top d = 0 \quad d^\top a - b^\top c = 1_n$$

Since $J^\top = -J = J^{-1}$, taking transpose and inverse of $g^\top J g = J$ (and rearranging a little) gives $gJg^o p = J$. Thus, an *equivalent* set of conditions for $g \in Sp_n$ is given by

$$ba^\top - ab^\top = 0 \quad dc^\top - cd^\top = 0 \quad da^\top - cb^\top = 1_n$$

The **standard maximal totally isotropic subspace** V_n of V is that spanned by the vectors $e_1, \ldots e_n$, where $\{e_i : i = 1, \ldots, 2n\}$ is the standard basis for k^{2n}. The (maximal proper) parabolic subgroup P stabilizing V_n is described in $n \times n$ blocks as

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

This is sometimes called the **Siegel parabolic** or **popular parabolic**. The **standard Levi component** M of P is

$$M = \{\begin{pmatrix} A & 0 \\ 0 & A^{\top -1} \end{pmatrix} : A \in GL_n(k)\}$$

where $A^{\top -1}$ means inverse of the transpose of A. The unipotent radical of P is

$$N = \{\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} : S = S^\top\}$$

where S can be any symmetric $n \times n$ matrix.

The *standard minimal parabolic* is

$$\{p = \begin{pmatrix} A & * \\ 0 & A^{\top -1} \end{pmatrix} : A \text{ is upper triangular}\}$$

This corresponds to the **standard maximal isotropic flag**

$$ke_1 \subset (ke_1 + ke_2) \subset (ke_1 + ke_2 + ke_3) \subset \ldots \subset (ke_1 + \ldots + ke_n)$$

Note that while the matrices in this parabolic subgroup have some sort of *upper-triangular* property, it is not *literally* so. Further, some of the zeros in the expression appear only because the matrix is required to lie in the symplectic group, not *just* because of stabilization of the indicated flag of subspaces.

The unipotent radical is the subgroup of such p having only 1's on the diagonal. The **standard Levi component** A of this minimal parabolic is the subgroup of all diagonal matrices of the form

$$\begin{pmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_n & & & \\ & & & a_1^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & a_n^{-1} \end{pmatrix}$$

Now we consider other maximal proper parabolics in Sp_n. Let V_ℓ be the subspace $ke_1 + \ldots + ke_\ell$ with $1 \le \ell < n$. The subgroup of Sp_n stabilizing V_ℓ must consist of matrices

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

where we use a decomposition into blocks of sizes

$$\begin{pmatrix} \ell\times\ell & * & * & * \\ * & (n-\ell)\times(n-\ell) & * & * \\ * & * & \ell\times\ell & * \\ * & * & * & (n-\ell)\times(n-\ell) \end{pmatrix}$$

(with compatible sizes off the diagonal), *and where there are further conditions which must be satisfied for the matrix to lie in* Sp_n. The **standard Levi component** consists of matrices with block decomposition

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & A^{\top-1} & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

with

$$A \in GL_\ell(k) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{n-\ell}(k)$$

The unipotent radical of this parabolic consists of elements of the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix}$$

with some relations among the entries. In particular, we have elements

$$\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x^\top & 1 \end{pmatrix}$$

which are not in the unipotent radical of the Siegel parabolic. The verification that this is the unipotent radical, with the definition as above in terms of geometric algebra, is just a computation.

All other *standard* parabolics are expressible as intersections of the standard maximal proper ones.

Now consider the *symplectic similitude group*

$$GSp_n = GSp_n(k) = \text{ similitude group of } \langle,\rangle \ =$$
$$= \{g \in GL_k(V) : \langle gu, gv\rangle = \nu(g)\langle u,v\rangle \ \ \forall u,v \in V, \text{ for some } \nu(g) \in k^\times\} =$$
$$= \{g \in GL_k(V) : g^\top J g = \nu(g)J, \text{ for some } \nu(g) \in k^\times\}$$

It is easy to check that the map $\nu : GSP_n \to k^\times$ is a group homomorphism, and that $\nu(g)^n = \det(g)$.

The *'shape'* of the standard parabolics, their unipotent radicals, and their standard Levi components is identical to that for Sp_n. A standard sort of element which is in GSp_n but not in Sp_n is

$$\begin{pmatrix} \lambda & & & & & \\ & \ddots & & & 0 & \\ & & \lambda & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

on which ν takes value $\lambda \in k^\times$. Note that these elements lie in the standard Levi component of every standard parabolic subgroup.

8.2 Orthogonal groups O(n,n) in coordinates

For the 'simplest' *orthogonal groups* $O(n,n)$ the discussion is *almost* identical to that for Sp_n, and we abbreviate it somewhat. The treatment of the similitude group $GO(n,n)$ parallels exactly that of GSp_n based upon Sp_n, as above.

We take $V = k^{2n}$, viewed as *column vectors*, and let

$$Q = \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix}$$

For $u, v \in V$, put

$$\langle u,v\rangle = v^\top Q v$$

This is a non-degenerate symmetric bilinear form on V, and makes V a *hyperbolic* space. The **standard orthogonal group** 'of signature (n,n)' is

$$O(n,n) = O(n,n)(k) = \text{ isometry group of } \langle,\rangle \ =$$
$$= \{g \in GL_k(V) : \langle gu, gv\rangle = \langle u,v\rangle \ \ \forall u,v \in V\} =$$

$$= \{g \in GL_{2n}(k) : g^\top Q g = Q\}$$

(It is a small exercise in linear algebra to check that the last condition is equivalent to the others).

Use $n \times n$ blocks in matrices. Upon multiplying out the condition $g^\top Q g = Q$, the condition for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be in $O(n,n)$ is

$$c^\top a + a^\top c = 0 \quad d^\top b + b^\top d = 0 \quad d^\top a + b^\top c = 1_n$$

An equivalent set of conditions for $g \in O(n,n)$ is given by

$$ba^\top + ab^\top = 0 \quad dc^\top + cd^\top = 0 \quad da^\top + cb^\top = 1_n$$

The **standard maximal totally isotropic subspace** V_n of V is that spanned by the vectors $e_1, \ldots e_n$, where $\{e_i : i = 1, \ldots, 2n\}$ is the standard basis for k^{2n}. The (maximal proper) parabolic subgroup P stabilizing V_n is described in $n \times n$ blocks as

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

This is sometimes called the **Siegel parabolic** or **popular parabolic**. The **standard Levi component** M of P is

$$M = \{\begin{pmatrix} A & 0 \\ 0 & A^{\top -1} \end{pmatrix} : A \in GL_n(k)\}$$

where $A^{\top -1}$ means inverse of the transpose of A. The unipotent radical of P is

$$N = \{\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} : S = -S^\top\}$$

where S can be any anti-symmetric $n \times n$ matrix.

Consider other maximal proper parabolics in $O(n,n)$: Let V_ℓ be the subspace $ke_1 + \ldots + ke_\ell$ with $1 \leq \ell < n$. The subgroup of $O(n,n)$ stabilizing V_ℓ must consist of matrices

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

with blocks of sizes

$$\begin{pmatrix} \ell \times \ell & * & * & * \\ * & (n-\ell) \times (n-\ell) & * & * \\ * & * & \ell \times \ell & * \\ * & * & * & (n-\ell) \times (n-\ell) \end{pmatrix}$$

(with compatible sizes off the diagonal), *with further conditions on the entries which must be met for the matrix to be in* $O(n,n)$. The **standard Levi**

component consists of matrices with block decomposition

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & A^{\top -1} & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

with

$$A \in GL_\ell(k) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(n-\ell, n-\ell)$$

The unipotent radical of this parabolic consists of elements of the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix}$$

with some relations among the entries. In particular, we have elements

$$\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x^\top & 1 \end{pmatrix}$$

which are not in the unipotent radical of the Siegel parabolic of $O(n, n)$.

8.3 Orthogonal groups O(p,q) in coordinates

Now we look at *certain* aspects of a somewhat more general type of orthogonal group. Fix integers $p \geq q \geq 0$, and put

$$Q = \begin{pmatrix} 0 & 0 & 1_q \\ 0 & 1_{p-q} & 0 \\ 1_q & 0 & 0 \end{pmatrix}$$

Then for *column vectors* $u, v \in V = k^{p+q}$ we define a non-degenerate symmetric bilinear form

$$\langle u, v \rangle = v^\top Q u$$

The orthogonal group of interest is the corresponding group

$$O(p,q) = O(p,q)(k) = \text{ isometry group of } \langle , \rangle =$$
$$= \{g \in GL_k(V) : \langle gu, gv \rangle = \langle u, v \rangle \ \ \forall u, v \in V\} =$$
$$= \{g \in GL_{p+q}(k) : g^\top Q g = Q\}$$

We note that, on other occasions, one might take the matrix Q associated with p, q to be

$$Q = \begin{pmatrix} 1_q & 0 & 0 \\ 0 & 1_{p-q} & 0 \\ 0 & 0 & -1_q \end{pmatrix}$$

instead. However, this choice of coordinates is suboptimal for our present purposes. Even the (straightforward) issue of getting from one coordinate version to another is not of great moment.

In the extreme case $q = 0$, one usually writes $n = p$, and

$$O(n) = \{g \in GL_n(k) : g^\top g = 1_n\}$$

We will not consider this case here, since the structures we wish to investigate (parabolic subgroups, etc.) are not visible in this choice of coordinates. In particular, unless we know much more about the nature of k, we have no idea whether there are any non-zero isotropic vectors.

For $(p+q) \times (p+q)$ matrices we use block decompositions of sizes

$$\begin{pmatrix} q \times q & * & * \\ * & (p-q) \times (p-q) & * \\ * & * & q \times q \end{pmatrix}$$

with corresponding sizes off the diagonal.

It is not particularly illuminating to write out the conditions on the nine blocks (in such decomposition) for a $(p+q) \times (p+q)$ matrix to be in the group $O(p,q)$. Rather, we will examine only the standard maximal proper parabolics, their unipotent radicals, and standard Levi components.

Let $\{e_i : 1 = 1, \ldots, p+q\}$ be the standard basis for k^{p+q}. The totally isotropic subspace V_q of V spanned by the vectors $e_1, \ldots e_q$ is *not* maximal isotropic, in general (since the quadratic form 1_{p-q} on k^{p-q} *may* have an isotropic vector). Nevertheless, we have a maximal proper parabolic subgroup P stabilizing V_q. In blocks as above, elements of P have the shape

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

with relations among the entries, which we now pursue by describing the unipotent radical and the standard Levi component. *Note indeed that the middle zero block on the bottom row is genuine, but depends upon the fact that the matrix is to lie in the orthogonal group.*

We claim that the unipotent radical R_uP of P consists of matrices of the form

$$\begin{pmatrix} 1 & x & S - \frac{1}{2}xx^\top \\ 0 & 1 & -x^\top \\ 0 & 0 & 1 \end{pmatrix}$$

where $S = -S^\top$ is $q \times Q$ skew-symmetric and x is arbitrary $q \times (p-q)$. That the general 'shape' should be as indicated is fairly clear. To see that the relations among the entries are as indicated, consider

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}^\top Q \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = Q$$

Upon multiplying out in terms of the blocks, we obtain

$$x + z^\top = 0 \quad y + y^\top + z^\top z = 0$$

as claimed.

The **standard Levi component** M of P consists of elements of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

with relations among the entries due to the fact that these elements must lie in $O(p,q)$. A straightforward computation of these relations shows that the Levi component is exactly all elements of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & A^{\top-1} \end{pmatrix}$$

where $A \in GL_q(k)$ is arbitrary and $\theta \in GL_{p-q}(k)$ must satisfy $\theta^\top\theta = 1$ (that is, θ is in another orthogonal group).

We can consider *certain* other maximal proper parabolics. Let V_ℓ be the subspace $ke_1 + \ldots + ke_\ell$ with $1 \leq \ell < q$. The subgroup of $O(p,q)$ stabilizing V_ℓ must consist of matrices with the shape

$$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{pmatrix}$$

where the blocks are of sizes

$$\begin{pmatrix} \ell\times\ell & * & * & * & * \\ * & (q-\ell)\times(q-\ell) & * & * & * \\ * & * & q\times q & * & * \\ * & * & * & \ell\times\ell & * \\ * & * & * & * & (q-\ell)\times(q-\ell) \end{pmatrix}$$

(with compatible sizes off the diagonal), *and certain relations among the entries must be satisfied for the matric to be in the orthogonal group.*

The **standard Levi component** consists of matrices with block decomposition

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 & h_{13} \\ 0 & h_{21} & h_{22} & 0 & h_{23} \\ 0 & 0 & 0 & A^{\top-1} & 0 \\ 0 & h_{31} & h_{32} & 0 & h_{33} \end{pmatrix}$$

with $A \in GL_\ell(k)$ and with

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

in the orthogonal group $O(p-\ell, q-\ell)$ attached to the symmetric bilinear form with matrix

$$\begin{pmatrix} 0 & 0 & 1_{q-\ell} \\ 0 & 1_{p-q} & 0 \\ 1_{q-\ell} & 0 & 0 \end{pmatrix}$$

8.4 Unitary groups in coordinates

Now we look at certain aspects of *unitary groups*.

Fix a quadratic field extension K of k with non-trivial automorphism σ of K over k. Fix integers $h, q > 0$, and fix a non-singular $h \times h$ matrix H satisfying $H^\sigma = H$, where $(H^\sigma)_{ij} = (H_{ji})^\sigma$. Put

$$Q = \begin{pmatrix} 0 & 0 & 1_q \\ 0 & H & 0 \\ 1_q & 0 & 0 \end{pmatrix}$$

Then for *column vectors* $u, v \in V = K^{h+2q}$ we define a non-degenerate σ-hermitian form

$$\langle u, v \rangle = v^\sigma Q u$$

where v^σ is the transpose of v with σ applied to every entry.

The unitary group of interest is the corresponding group

$$U(Q) = \text{ isometry group of } \langle , \rangle \ =$$
$$= \{g \in GL_K(V) : \langle gu, gv \rangle = \langle u, v \rangle \ \ \forall u, v \in V\} =$$
$$= \{g \in GL_{h+2q}(K) : g^\sigma Q g = Q\}$$

In the extreme case $q = 0$, one usually writes

$$U(H) = \{g \in GL_h(K) : g^\sigma H g = H\}$$

We will not consider this case here, since parabolic subgroups are not visible in this choice of coordinates. In particular, unless we know much more about the nature of Q, K, k, we have no idea whether there are any non-zero isotropic vectors.

For $(h+2q) \times (h+2q)$ matrices we use block decompositions of sizes

$$\begin{pmatrix} q \times q & * & * \\ * & h \times h & * \\ * & * & q \times q \end{pmatrix}$$

with corresponding sizes off the diagonal.

As with the more general orthogonal groups, it is not particularly illuminating to write out the conditions on the nine blocks in such decomposition.

Let e_i be the standard basis elements for k^{h+2q}. The totally isotropic subspace V_q of V spanned by the vectors $e_1, \ldots e_q$ may *not* maximal isotropic, in general. Nevertheless, we have a maximal proper parabolic subgroup P stabilizing V_q. In blocks as above, elements of P have the shape

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

with relations among the entries, which we now explain by describing the unipotent radical and the standard Levi component.

We claim that the unipotent radical R_uP of P consists of matrices of the form

$$\begin{pmatrix} 1 & -z^\sigma H & S - \frac{1}{2}z^\sigma Hz \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $S = -S^\sigma$ is arbitrary $q \times Q$ skew-hermitian and z is arbitrary $h \times q$ with entries in K. That the general 'shape' should be

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

as indicated is fairly clear. To see that the relations among the entries are as indicated, consider

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}^\sigma Q \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = Q$$

Upon multiplying out in terms of the blocks, we obtain

$$x + z^\sigma H = 0 \quad y + y^\sigma + z^\sigma Hz = 0$$

as claimed.

The **standard Levi component** M of P consists of elements of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

with relations among the entries due to the fact that these elements must lie in the unitary group. Computation shows that the Levi component is exactly all elements of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & A^{\sigma-1} \end{pmatrix}$$

where $A \in GL_q(K)$ is arbitrary and θ lies in the smaller unitary group $U(H)$.

9. Spherical Construction for GL(n)

- Construction of the spherical building for GL(n)
- Verification of the building axioms
- The action of GL(n) on the spherical building
- The spherical BN-pair in GL(n)
- Analogous treatment of SL(n)
- The symmetric group as Coxeter group

Using notions defined earlier in our general discussion (3.1) of chamber complexes, we describe an *incidence geometry* from which we obtain a *flag complex* which is a *thick building* (4.1), whose associated *BN-pair* (5.2) has parabolics (5.3) which really are the parabolic subgroups of $GL(n)$ in the geometric algebra sense discussed above in (7.1). This will be a building of **type** A_{n-1}, in the sense that the apartments are Coxeter complexes (3.4) of type A_{n-1}, where the latter data is as given in (2.2).

Among other things, we will see that the apartments are Coxeter complexes attached to the Coxeter system (W, S), where W is the symmetric group on $\{1, 2, \ldots, n\}$ and S consists of *adjacent transpositions* σ_i for $i = 1, \ldots, n-1$. (That is, s_i interchanges i and $i+1$ and leaves unchanged all others). It is certainly not clear *a priori* that the symmetric group is a Coxeter group, etc. However, granting that this (W, S) is a Coxeter system, the Coxeter data is visible: if $|i-j| \neq 1$, then s_i and s_j commute; on the other hand, $s_i s_{i+1}$ is a 3-cycle, so is of order 3. *This is the Coxeter system of type* A_{n-1}.

The first section constructs the thick building, while the second section verifies the necessary properties of a building. Since the apartments are *finite* complexes, they are said to be *spherical*, as is the building.

Then we check that $GL(n)$ acts strongly transitively on this building, that is, is transitive on the set of pairs (C, A) where C is a chamber contained in an apartment A. Last, we explicitly identify the BN-pair that arises, seeing that the 'B' really is a minimal parabolic in the geometric algebra sense.

Incidentally, we have already shown that in a *spherical* building there is a unique apartment system. In particular, any apartment system we construct is unavoidably the *maximal* one. Thus, while it might appear that we can exercise volition here, we in fact *cannot*, in this regard.

9.1 Construction of the spherical building for GL(n)

We construct buildings whose apartments are of type A_{n-1}.

Let k be a field, and V an n-dimensional vectorspace over k. Let $G = GL_k(V)$ be the group of k-linear automorphisms of V. We may often write simply $GL(n)$ or $GL(V)$ for this group. (All this works as well for vectorspaces over division rings, too, but we won't worry about this).

Let Ξ be the set of proper, non-trivial vector subspaces of V (that is, subspaces x which are neither $\{0\}$ nor V). We have an incidence relation $\sim$ on Ξ defined as follows: write $x \sim y$ for $x, y \in \Xi$ if either $x \subset y$ or $y \subset x$.

As defined earlier (3.1), the associated *flag complex* X is the simplicial complex with vertices Ξ and simplices which are *mutually incident* subsets of Ξ. That is, the simplices in X are subsets σ of Ξ so that, for all $x, y \in \sigma$, $x \sim y$. Thus, in this example, the flags in an incidence geometry are the same things as flags of subspaces of a vector space, as in (7.1).

The maximal simplices in X are in bijection with sequences (maximal flags)

$$V_1 \subset V_2 \subset \ldots \subset V_{n-1}$$

of subspaces V_i of V where V_i is of dimension i.

In the present context, a **frame** in V is an unordered n-tuple $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ of *lines* (one-dimensional subspaces) λ_i in V so that

$$\lambda_1 \oplus \ldots \oplus \lambda_n \approx \lambda_1 + \ldots + \lambda_n = V$$

We take a set $\mathcal{A}$ of subcomplexes indexed by frames $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ in V: the associated apartment $A = A_{\mathcal{F}} \in \mathcal{A}$ consists of all simplices σ with vertices which are subspaces ξ expressible as

$$\xi = \lambda_{i_1} \oplus \ldots \oplus \lambda_{i_m}$$

for some m-tuple $i_1, \ldots, i_m$.

9.2 Verification of the building axioms

Keep all assumptions and notation of the previou section. Now we verify the conditions (4.1) for a simplicial complex to be a thick building, and at the end check the *type-preserving strong transitivity* (5.2) of the group action.

The facets F_j of a maximal simplex

$$C = (V_1 \subset \ldots \subset V_{n-1})$$

as above are in bijection with indices $1 \leq j \leq n-1$, by omitting the j^{th} subspace. That is, the j^{th} facet is

$$F_j = (V_1 \subset \ldots \subset V_{j-1} \subset V_{j+1} \subset \ldots \subset V_{n-1})$$

The other maximal simplices in X with facet F_j are flags

$$V_1 \subset \ldots \subset V_{j-1} \subset V_j' \subset V_{j+1} \subset \ldots \subset V_{n-1}$$

where, pointedly, the only change is at the j^{th} spot.

It is important to note that *maximal simplices in the apartment A are in bijection with choices of ordering of the lines $\lambda_1, \ldots, \lambda_n$*: to an ordering $\lambda_{i_1}, \ldots, \lambda_{i_n}$ we associate the maximal set of mutually incident subspaces

$$V_j = \lambda_{i_1} \oplus \ldots \oplus \lambda_{i_j}$$

(and the corresponding maximal flag). We use this in what follows.

First we prove that each simplicial complex $A \in \mathcal{A}$ is a *thin chamber complex*. Fix a frame $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ specifying A.

For each index j, one must ascertain the j-dimensional subspaces V_j' *within the apartment A*, so that

$$V_{j-1} \subset V_j' \subset V_{j+1}$$

and so that the subspace V_j' is a direct sum of some of the lines λ_i. On one hand, the requirement $V_j' \subset V_{j+1}$ implies that the direct sum expression for V_j' is constrained to merely omit one of the lines in the sum expressing V_{j+1}. On the other hand, the requirement $V_j' \supset V_{j-1}$ implies that the direct sum expression for V_j' *cannot* omit the lines in the sum expressing V_{j-1}. Thus, the only choice remaining to describe V_j' is the choice of which of the two lines $\lambda_{i_j}, \lambda_{i_{j+1}}$ to exclude.

As noted just above, the maximal simplices in A are in bijection with orderings of the lines in the frame. The previous paragraph shows that *the effect of moving across the j^{th} face is to interchange λ_{i_j} and $\lambda_{i_{j+1}}$ in this ordering*. That is, the ordering corresponding to the *other* chamber with the same j^{th} face is obtained by interchanging i_j and i_{j+1}.

Thus, to prove that each apartment A is indeed a chamber complex, we must find a gallery from the maximal simplex specified by the ordering of lines

$$\lambda_1, \ldots, \lambda_n$$

to a maximal simplex

$$\lambda_{i_1}, \ldots, \lambda_{i_n}$$

for an arbitrary permutation of the indices. We have noted that movement across the j^{th} facet interchanges the j^{th} and $(j+1)^{\text{th}}$ lines in such an ordering. *Since the permutation group on n things is generated by adjacent transpositions $(j, j+1)$, there is a gallery connecting any two chambers in the apartment*. Note here that we only use the *generation by* adjacent transpositions, and nothing more delicate.

Incidentally, above we saw above that there are only two choices (inside an apartment A) for a j-dimensional subspace containing a given V_{j-1} and contained in a given V_{j+1}, since the choice of this subspace is just a choice between two lines. Thus, the apartment A is *thin*, as asserted.

Now we address the *thickness* of the whole complex. Given $(j-1)$-dimensional and $(j+1)$-dimensional subspaces V_{j-1}, V_{j+1} in V, the choice of a j-dimensional subspace V_j' between them *unconstrained by restriction to an apartment* is equivalent to choice of a *line* in the quotient

$$V_{j+1}/V_{j-1} \approx k^2$$

where k is the underlying field. If k is infinite then there certainly are infinitely-many distinct lines in this space. If k has finite cardinality q, then there are

$$(q^2-1)/(q-1) = q+1 \geq 2+1 = 3$$

distinct lines. Thus, the whole flag complex is *thick*.

Now we show that any two maximal simplices in X lie inside one of the subcomplexes $A \in \mathcal{A}$. This, together with the fact (already proven) that the subcomplexes $A \in \mathcal{A}$ are *chamber* complexes, will also prove that the whole complex X is a chamber complex (shown to be *thick* in the previous paragraph). apartment. That is, given two maximal flags

$$V_1 \subset \ldots \subset V_{n-1}$$

$$V_1' \subset \ldots \subset V_{n-1}'$$

we must find a frame $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ so that all the V_i and all the V_i' are sums of the λ_i. To this end we reprove a quantitative version of a Jordan-Holder -type theorem:

We view the two flags as giving *composition series* for V. Then for each i, we have a *filtration* of V_i/V_{i-1} given by the V_j':

$$\frac{(V_i \cap V_o') + V_{i-1}}{V_{i-1}} \subset \frac{(V_i \cap V_1') + V_{i-1}}{V_{i-1}} \subset \ldots \subset \frac{(V_i \cap V_n') + V_{i-1}}{V_{i-1}}$$

For all indices i, j we have

$$\frac{V_i}{V_{i-1}} \supsetneq \frac{(V_i \cap V_j') + V_{i-1}}{V_{i-1}} \xrightarrow{\text{onto}}$$

$$\xrightarrow{\text{onto}} \frac{(V_i \cap V_j') + V_{i-1}}{V_{i-1} + (V_i \cap V_{j-1}')} \approx \frac{V_i \cap V_j'}{(V_{i-1} \cap V_j') + (V_i \cap V_{j-1}')}$$

The space V_i/V_{i-1} is one-dimensional, so for given i there is exactly one index j for which the quotient

$$\frac{(V_i \cap V_j') + V_{i-1}}{V_{i-1}}$$

is one-dimensional. With this j, we claim that

$$V_i \cap V_{j-1}' \subset V_{i-1}$$

If not, then

$$V_i = V_{i-1} + (V_i \cap V_{j-1}')$$

since the dimension of V_i is just one greater than that of V_{i-1}. But by its definition, j is the smallest among indices ℓ so that

$$V_i = V_{i-1} + (V_i \cap V'_\ell)$$

Thus, the claim is proven. Thus, given i, there is exactly one index j for which

$$\frac{V_i \cap V'_j}{(V_{i-1} \cap V'_j) + (V_i \cap V'_{j-1})}$$

is one-dimensional. The latter expression is symmetrical in i and j, so there is a permutation π so that this expression is one-dimensional only if $j = \pi(i)$, otherwise is 0.

By symmetry, with i, j related by the permutation π, we have isomorphisms

$$\frac{V_i}{V_{i-1}} \approx \frac{V_i \cap V'_j}{(V_{i-1} + (V_i \cap V'_{j-1})) \cap V'_j} \approx$$

$$\approx \frac{V_i \cap V'_j}{(V_{i-1} \cap V'_j) + (V_i \cap V'_{j-1})} \approx \frac{V'_j}{V'_{j-1}}$$

Given the previous, choose a line λ_i lying in $V_i \cap V'_j$ which maps *onto* the one-dimensional quotients. The collection of such lines provides the desired *frame* specifying an apartment containing *both* chambers.

To complete the verification that we have a thick building, we must show that, if a chamber C and a simplex x both lie in two apartments A, B, then there is a chamber-complex isomorphism $f : B \to A$ fixing both x and C pointwise. We will in fact give the map by giving a bijection between the lines in the respective frames: this surely would give a face-relation-preserving bijection between the simplices. And we will prove, instead, the apparently stronger assertion that, given two apartments A, B containing a chamber C, there is an isomorphism $f : B \to A$ fixing $A \cap B$ pointwise.

Let $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ and $\mathcal{G} = \{\mu_1, \ldots, \mu_n\}$ be the frames specifying the apartments A, B, respectively. Without loss of generality, we can renumber the lines in both of these so that the chamber C corresponds to the orderings

$$(\lambda_1, \ldots, \lambda_n) \text{ and } (\mu_1, \ldots, \mu_n)$$

That is, the i-dimensional subspace occurring as a vertex of C is

$$\lambda_1 + \ldots + \lambda_i = \mu_1 + \ldots + \mu_i$$

Consider the map

$$f : B \to A$$

given on vertices by

$$\lambda_{i_1} + \ldots + \lambda_{i_m} \to \mu_{i_1} + \ldots + \mu_{i_m}$$

for any m distinct indices $i_1 < \ldots < i_m$. Anticipating that the Uniqueness Lemma would imply that there is at most one such map, this must be it.

To show that f is the identity on $A \cap B$ it suffices to show that it is the identity on all 0-simplices in the intersection. If a 0-simplex x lies in $A \cap B$ then x is a subspace of V which can be written as a sum of some of the λ_i and also as a sum of some of the μ_i. What we will show is that, if

$$\lambda_{i_1} + \ldots + \lambda_{i_m} = \mu_{j_1} + \ldots + \mu_{j_m}$$

then

$$i_1 = j_1, \ \ i_2 = j_2, \ \ldots, \ i_m = j_m$$

The later equalities then would assure that all of $A \cap B$ would be fixed pointwise by f.

Suppose that we have a subspace x (a 0-simplex) in $A \cap B$ given as

$$x = \lambda_{i_1} + \ldots + \lambda_{i_m} = \mu_{j_1} + \ldots + \mu_{j_m}$$

Suppose that it is *not* the case that $i_\nu = j_\nu$ for all ν: let ν be the largest (with $1 \leq \nu \leq m$) so that $i_\nu \neq j_\nu$. Without loss of generality (by symmetry), suppose that $i_\nu < j_\nu$. By hypothesis, we have

$$\lambda_1 + \lambda_2 + \ldots + \lambda_{j_\nu - 2} + \lambda_{j_\nu - 1} = \mu_1 + \mu_2 + \ldots + \mu_{j_\nu - 2} + \mu_{j_\nu - 1}$$

Summing this subspace with x, we obtain

$$\lambda_1 + \lambda_2 + \ldots + \lambda_{j_\nu - 1} + \lambda_{i_{\nu+1}} + \ldots + \lambda_{i_m} =$$

$$= \mu_1 + \mu_2 + \ldots + \mu_{j_\nu - 1} + \mu_{j_\nu} + \mu_{j_{\nu+1}} + \ldots + \mu_{j_m}$$

But the left-hand side has dimension

$$(j_\nu - 1) + (m - \nu) = m + j_\nu - \nu - 1$$

while the right-hand side has dimension

$$(j_\nu - 1) + (m - \nu + 1) = m + j_\nu - \nu$$

This is impossible, so it must have been that $i_\nu = j_\nu$ for all ν. This proves the second axiom for a building.

Thus, the complex constructed by taking flags of subspaces is a thick building, with apartment system given via frames, which themselves are decompositions of the whole space as direct sums of lines.

9.3 Action of GL(n) on the spherical building

The previous section proves that we have a thick building, which is said to be of type A_{n-1} since its apartments are Coxeter complexes (3.4) of that type (2.1). Now we need but a little further effort to check that $GL(V)$ acts strongly transitively (5.2) and preserves types on this building.

First, although we know (4.4) that there *exists* an essentially unique labeling on this building, a tangible labeling is available and is more helpful. By the uniqueness, our choice of description of the labeling is of no consequence. So the following intuitively appealing labeling is perfectly fine for our purposes.

To determine the type of a simplex in X, we need only determine the type of its vertices. In the present example, we define the *type* of a vertex to be the dimension of the corresponding subspace, thereby defining a typing on all simplices. The action of $GL(V)$ is certainly type-preserving.

Given two apartments specified by two frames

$$\mathcal{F} = \lambda_1, \ldots, \lambda_n$$

$$\mathcal{F}' = \lambda_1', \ldots, \lambda_n'$$

there is $g \in GL(V)$ so that $g\lambda_i = \lambda_i'$. That is, $GL(V)$ is transitive on apartments. And it is immediate that the action of $GL(V)$ sends apartments to apartments.

The chambers within an apartment A specified by a frame

$$\mathcal{F} = \lambda_1, \ldots, \lambda_n$$

are in bijection with choices of ordering of the lines λ_i. From the previous paragraph, we observe that the stabilizer of an apartment is the group of linear maps which stabilize the set of lines making up the frame. This certainly includes linear maps to give arbitrary permutations of the set of lines in the frame. That is, we see that the stabilizer of an apartment is transitive on the chambers within it. This, together with the previous paragraph, shows that $GL(V)$ is indeed *strongly* transitive on the building, that is, is transitive on the set of pairs (C, A) where C is a chamber contained in an apartment A.

This completes the verification that $GL(V)$ acts strongly transitively upon the spherical building constructed in the previous section, and preserves types.

9.4 The spherical BN-pair in GL(n)

We emphasize that the subgroups B (stabilizers of chambers) in the BN-pairs arising from the action of $GL(n)$ on the thick building above really are minimal parabolic subgroups in the geometric algebra sense (7.1). Indeed, the construction of this building of type A_{n-1} was guided exactly by the aim to have this happen. Thus, *facts about parabolic subgroups appear as corollaries to discussion of buildings and BN-pairs.*

Repeating: by the very definition of this building, stabilizers of simplices in the building are stabilizers of flags of subspaces. Thus, in particular, the minimal parabolic subgroups of $GL(n)$ really are obtained as stabilizers of chambers of this thick building.

Remarks: We could reasonably assert that the collection of all chambers in the spherical building is in natural bijection with the collection of all minimal parabolic subgroups in $GL(n)$. More broadly, *the collection of simplices in the building is in bijection with all parabolics in the group, and the face relation is inclusion reversed.* Or, we could say that the set of *vertices* was the collection of *maximal* proper parabolic subgroups, and that a collection of such gave a simplex if and only if their intersection were again a parabolic subgroup.

Remarks: If we were to comply with the terminology of algebraic groups, then we would have to say that this B is *the group of k-valued points of* a minimal k-parabolic. We will *not* worry about adherence to this orthodoxy.

Let $e_1, \ldots, e_n$ be the standard basis for the n-dimensional k-vectorspace $V = k^n$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots$$

The *standard frame* consists of the collection of lines ke_i. The *standard maximal flag* is

$$V_1 = ke_1 \subset V_2 = ke_1 \oplus ke_2 \subset \ldots \subset V_{n-1} = ke_1 \oplus \ldots \oplus ke_n$$

By definition, the B in the BN-pair is the stabilizer B in $GL(n, k)$ of this flag: writing the vectors as column vectors, we find that B consists of *upper triangular matrices*

$$\begin{pmatrix} * & \ldots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$$

This is indeed a *Borel subgroup*, that is, a *minimal parabolic subgroup.*

As described in the previous section, the apartment A_o corresponding to the standard frame has simplices whose vertices are subspaces expressible as sums of these one-dimensional subspaces. It is elementary that the stabilizer $\mathcal{N}$ of this frame consists of *monomial matrices*, that is, matrices with just one non-zero entry in each row and column. The

Then the subgroup T here is

$$T = B \cap \mathcal{N} = \text{ upper-triangular monomial matrices } =$$

$$= \text{ diagonal matrices}$$

Then the *Weyl group* W (which we have shown indirectly to be a Coxeter group) is

$$W = \mathcal{N}/T \approx \; n \times n \text{ permutation matrices } \approx S_n$$

where S_n is the permutation group on n things.

It is important to note that, while the group W is not defined to be a subgroup of $G = GL(n, k)$, *in this example* it has a *set of representatives* which do form a subgroup of G.

In this example, the Bruhat-Tits decomposition (5.1), (5.4) can be put in more prosaic terms: *every non-singular $n \times n$ matrix over a field can be* *written as a product*

$$\text{upper-triangular} \times \text{permutation} \times \text{upper-triangular}$$

This is not so hard to prove by hand.. Indeed, one can prove directly the further fact (following from Bruhat-Tits) that the permutation matrix which enters is uniquely determined.

Remarks: The finer details of the BN-pair and Bruhat-Tits decomposition properties are not easy to see directly. The cell multiplication rules are inexplicable without explicit accounting for the Coxeter system. And, for example, the fact that any subgroup of $GL(n, k)$ containing the upper triangular matrices B is necessarily a (standard) parabolic is not clear.

More can be said. In any case, we have successfully recovered a refined version of seemingly elementary facts about $GL(n)$ as by-products of the construction of the spherical building and the corresponding BN-pair.

9.5 Analogous treatment of SL(n)

Here we make just a few remarks to make clear that the strongly transitive label-preserving action of $GL(n)$ on the thick building of type A_{n-1} constructed above, when restricted to an action of $SL(n)$, *is still strongly transitive*. Thus, the BN-pair obtained for $GL(n)$ has an obvious counterpart for $SL(n)$.

Certainly $SL(n)$ preserves the labels, since it is a subgroup of $GL(n)$ and $GL(n)$ preserves labels. To prove that $SL(n)$ is strongly transitive, it *suffices* to show that

$$T \cdot SL(n) = GL(n)$$

where T is the stabilizer in $GL(n)$ of an apartment A and simultaneously of a chamber C within A. Indeed, quite generally, if G is a group acting transitively on a set $\mathcal{X}$, and if H is a subgroup of G, and if $G = \Theta H$ where Θ is the isotropy group of a point in $\mathcal{X}$, then H is also transitive on the set. In the present situation, we can easily arrange choice of A and C so that T is the subgroup of all diagonal matrices in $GL(n)$. But of course every element g of $GL(n)$ can be written as

$$g = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \det g \end{pmatrix} \left(\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \det g^{-1} \end{pmatrix} g \right)$$

This expresses g as a product of an element of T and an element of $SL(n)$, as desired.

In fact, from this discussion we see that for *any* group G with

$$SL(n) \subset G \subset GL(n)$$

we can obtain a corresponding BN-pair and all that goes with it. Of course, for *smaller* groups inside $SL(n)$ we cannot expect these properties to remain.

9.6 The symmetric group as Coxeter group

Incidental to all this is that we have given a somewhat circuitous proof of the fact that symmetric groups S_n are Coxeter groups, generated by *adjacent transpositions*

$$\alpha_j = (j, j+1)$$

It is clear that the 3-cycle $\alpha_j \alpha_{j+1}$ obtained has order 3, and that $\alpha_i \alpha_j = \alpha_j \alpha_i$ if $|i-j| \neq 1$. This is the **Coxeter system of type** A_{n-1}.

But without invoking all the result above it is *not* entirely clear that there are no other relations. Our discussion of $GL(n)$ gives an indirect proof of this.

We recall the basic idea of the proof that this is a Coxeter system: we constructed a thick building, whose apartments are (from general results) Coxeter complexes. And, in verifying the building axioms, via our identification of chambers with orderings of $\{1, \ldots, n\}$, we noted *reflection* through the i^{th} facet has the effect of interchanging the i^{th} and $(i+1)^{\text{th}}$ items in the ordering. Thus, by these identification, the Coxeter system (W, S) so obtained really does give $W = S_n$ and S the set of adjacent transpositions.

So we can apply the general theorems about Coxeter groups to the symmetric group. Some of these conclusions are easy to reach without this general machinery, but many are not so trivial. Since such results are not needed in the sequel, we leave this investigation to the interested reader.

In particular, it is of some interest to verify that

$$w_\ell = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \ldots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

is the *longest* element in this Coxeter group. This is best proven by identifying the roots, and examining the action of permutation matrices upon them.

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10. Spherical Construction for Isometry Groups

- Construction of spherical buildings for isometry groups
- Verification of the building axioms
- The action of the isometry group
- The spherical BN-pair in isometry groups
- Analogues for similitude groups and special groups

Now we carry out the natural construction of a thick building for all isometry groups (7.2) with the *exception* of certain orthogonal groups $O(n, n)$, which require a different treatment given in the next section. All other types of orthogonal groups, symplectic groups, and unitary groups are covered by the present discussion. The present construction does give a 'building' even for $O(n, n)$, but it fails to be *thick*, which complicates application of general results.

Most of the discussion will strongly resemble that for $GL(n)$. There are substantial simplifications possible if one specializes to the case of symplectic groups, that is, non-degenerate *alternating* forms. One might execute such simplifications an an illuminating exercise.

As in the previous discussion of $GL(V)$, we will construct buildings whose apartments are *finite* complexes. Thus, these complexes and the building as a whole are *spherical*. And recall that we have shown that in a spherical building there is a unique (hence, *maximal*) apartment system.

10.1 Construction of spherical buildings for isometry groups

Here we construct (spherical) buildings of type C_n (2.1).

Fix a field k. Let D be a division ring with involution σ, and suppose that k is the collection of elements in the center of D which are fixed by the involution.

Let V be a finite-dimensional D-vectorspace with a non-degenerate form $\langle , \rangle$ with the properties

$$\langle \alpha u, v + v' \rangle = \langle u, \rangle v\alpha + \langle u, v' \rangle \alpha'$$

$$\langle u, v \rangle = \epsilon \langle v, u \rangle^\sigma$$

for a fixed $\epsilon \in \{\pm 1\}$, for all $u, v \in V$, and for all $\alpha \in D$. Let G be the *isometry group* of V with the form $\langle , \rangle$:

$$G = \{g \in GL_D(V) : \langle gu, gv \rangle = \langle u, v \rangle \ \ \forall u, v\}$$

As was done with $GL(n)$ earlier (2.3), now for an isometry group we describe an *incidence geometry* from which we obtain a *flag complex* which is a *thick building*, whose associated BN-pair has *parabolics* which really are the parabolic subgroups of G.

We suppose that the largest *totally isotropic* subspace of V has D-dimension n. By Witt's theorem (7.3), from geometric algebra, this is the common dimension of all maximal totally isotropic subspaces.

Let Ξ be the collection of non-zero *totally isotropic* D-subspaces of V. Recall that a subspace V' of V is said to be totally isotropic if $\langle u, v\rangle = 0$ for all $u, v \in V'$. We define an *incidence relation* $\sim$ on Ξ by writing $x \sim y$ if *either* $x \subset y$ *or* $y \subset x$.

The associated *flag complex* X is the simplicial complex with vertex set Ξ and simplices which are *mutually incident* subsets of Ξ. That is, the simplices of X are subsets σ of Ξ so that for all $x, y \in \sigma$ we have $x \sim y$. The *maximal simplices* in X are the maximal flags

$$V_1 \subset \ldots \subset V_n$$

of totally isotropic subspaces V_i of V, where the dimension of V_i is i.

A **frame** $\mathcal{F}$ in the present setting is an unordered $2n$-tuple of lines (one-dimensional D-subspaces) in V, which admit grouping into unordered pairs $\lambda_i^{+1}, \lambda_i^{-1}$ whose sums $H_i = \lambda_i^{+1} + \lambda_i^{-1}$ are *hyperbolic planes* H_i (in the sense of geometric algebra) in V, and so that

$$H_1 + \ldots + H_n$$

is an *orthogonal* direct sum.

We consider the set $\mathcal{A}$ (the anticipated apartment system) of subcomplexes A of X indexed by frames $\mathcal{F}$ in the following manner: the associated subcomplex $A_{\mathcal{F}}$ (anticipated to be an apartment) consists of all simplices σ with all vertices being totally isotropic subspaces ξ expressible as

$$\xi = \lambda_{i_1}^{\epsilon_1} \oplus \ldots \oplus \lambda_{i_d}^{\epsilon_d}$$

for some unordered d-tuple $\{i_1, \ldots, i_d\}$, where for each i the ϵ_i is ± 1.

10.2 Verification of the building axioms

Keep the notation of the previous section.

The facets F_i of a maximal simplex

$$C = (V_1 \subset \ldots \subset V_n)$$

are in bijection with indices $1 \le i \le n$, by omitting the i^{th} subspace in the flag. The other maximal simplices in X with facet F_i correspond to flags

$$V_1 \subset \ldots \subset V_{i-1} \subset V_i' \subset V_{i+1} \subset \ldots \subset V_n$$

where the only allowed change is at the i^{th} spot.

We note that maximal simplices in an apartment A corresponding to the frame $\mathcal{F} = \{\lambda_i^{\pm 1}\}$ as above are in bijection with choices of orderings of the hyperbolic planes $H_i = \lambda_i^{+1} + \lambda_i^{-1}$ and (further) choice of one of the two distinguished lines from each hyperbolic plane, as follows: to a choice $\lambda_{i_1}^{\epsilon_1}, \ldots, \lambda_{i_n}^{\epsilon_n}$ we associate the totally isotropic subspaces

$$V_j = \lambda_{i_1} \oplus \ldots \oplus \lambda_{i_j}$$

and the flag

$$V_1 \subset \ldots \subset V_n$$

This bijection is useful in what follows.

First, we prove that each simplicial complex $A \in \mathcal{A}$ really is a *thin chamber complex*. Fix a frame $\mathcal{F}$ with $H_i = \lambda_i^{+1} \oplus \lambda_i^{-1}$ as above, specifying A. We consider the maximal isotropic flag

$$V_1 \subset \ldots \subset V_n$$

with

$$V_n = \lambda_1^{\epsilon_1} \oplus \lambda_2^{\epsilon_2} \oplus \ldots \oplus \lambda_n^{\epsilon_n}$$

with fixed choice of superscripts $\epsilon_i \in \{\pm 1\}$.

For each index $i < n$, we must ascertain the possibilities for an i-dimensional subspace V_i' in A so that

$$V_{i-1} \subset V_i' \subset V_{i+1}$$

and so that V_i' is a direct sum of the lines λ_i (in order for it to belong in the apartment A). (The case $i = n$ requires separate treatment).

On one hand, the requirement $V_i' \subset V_{i+1}$ implies that the direct sum expression for V_i' is obtained by omitting one of the lines from the direct sum expression for V_{i+1}. On the other hand, the requirement $V_{i-1} \subset V_i'$ implies that the expression for V_i' cannot omit any of the lines expressing V_{i-1}. Thus, the only choice involved in specifying V_i is the choice of whether to omit $\lambda_{j_i}^{\epsilon_1}$ or $\lambda_{j_{i+1}}^{\epsilon_{i+1}}$ from the expression

$$V_{i+1} = \lambda_{i_1}^{\epsilon_1} \oplus \ldots \oplus \lambda_{j_i}^{\epsilon_i} \oplus \lambda_{j_{i+1}}^{\epsilon_{i+1}}$$

in the case that $i < n$.

If $i = n$, then we are concerned with choices for V_n', and the constraints are that $V_{n-1} \subset V_n'$(and that V_n' be totally isotropic. In addition to the original V_n, the only other choice inside the subcomplex A would be to replace $\lambda_n^{\epsilon_n}$ by the *other* line inside H_n, that is, by $\lambda_n^{-\epsilon_n}$.

Keeping in mind the identification of maximal simplices in A with orderings of the hyperbolic planes together with choice of line within each plane, we can paraphrase the observations of the last paragraph as asserting that *the effect of moving across the i^{th} facet is to interchange the i^{th} and $(i+1)^{\text{th}}$ hyperbolic planes if $i < n$, and exchanges the lines in the n^{th} plane if $i = n$*. That is, more symbolically, moving across the i^{th} facet exchanges H_{j_i} and $H_{j_{i+1}}$ if $i < n$, and exchanges $\lambda_n^{\epsilon_n}$ and $\lambda_n^{-\epsilon_n}$ in the case $i = n$.

We need to describe the **signed permutation group** on n things in order to finish the proof that the apartments are thin chamber complexes. Incidentally, this will identify in more elementary terms the Coxeter group obtained here. Let S_n be the permutation group on n things, and let H be the direct sum of n copies of the group $\{\pm 1\}$. Let $\pi \in S_n$ act on H by

$$(\epsilon_1, \ldots, \epsilon_n)^\pi = (\epsilon_{\pi(1)}, \ldots, \epsilon_{\pi(n)})$$

Then we can form the semi-direct product

$$S_n^\pm = H \times \lhd S_n$$

This is the **signed permutation group** on n things.

To prove that the subcomplex A is a *chamber* complex, by definition we must find a gallery connecting any two maximal simplices. By the previous discussion, this amounts to showing that *the adjacent transpositions* $(i \;\; i+1)$ *together with*

$$(+1, +1, \ldots, +1, -1) \in H$$

(the change-sign just at the n^{th} *place) generate the signed permutation group on* n *things.* This is an elementary exercise.

Incidental to the above we did observe that there were always exactly *two* choices for maximal simplices (inside A) with a given facet. Thus, indeed, these *apartments are thin chamber complexes.*

Now we consider the issue of the *thickness* of the whole complex. *This argument would fail for an orthogonal group* $O(n,n)$.

In the context of the discussion above, for $i < n$, given totally isotropic subspaces $V_{i-1} \subset V_{i+1}$ we must show that there are at least 3 possibilities for V_i with $V_{i-1} \subset V_i \subset V_{i+1}$. In the case $i = n$, the issue is to show that for given totally isotropic V_{n-1} there are at least 3 choices for totally isotropic V_n so that $V_{n-1} \subset V_n$. For $i < n$, the choice is equivalent to the choice of a line in the two-dimensional D-vectorspace V_{i+1}/V_{i-1}, and for $i = n$ the choice is that of an *isotropic* line in the two-dimensional vectorspace $V_{n-1}^\perp/V_{n-1}$ with its natural non-degenerate form.

If the ring D is *infinite*, we certainly have 3 or more lines in a two-dimensional vectorspace. If D is finite with q elements, then the number of lines in a two-dimensional vectorspace is

$$(q^2 - 1)/(q - 1) = q + 1 > 2$$

Now we come to the delicate issue of the number of *isotropic* lines in $V_{n-1}^\perp/V_{n-1}$. By elementary geometric algebra, this space can be written as $H \oplus Q$ where H is a hyperbolic plane and Q is *anisotropic.* At this point we must consider various possibilities separately.

First, and most simply, if we have an *alternating space*, that is, if $D = k$, σ is trivial, and $\epsilon = -1$, then there *are* no non-trivial anisotropic spaces, and in fact any one-dimensional subspace is isotropic (as long as the characteristic is not 2). Thus, to check thickness in this case we suppose that the field k has q

elements, and count the number of lines in a two-dimensional k-vectorspace: it is

$$(q^2 - 1)/(q - 1) = q + 1 \geq 2 + 1 = 3$$

so we have thickness in this case.

Second, we consider *symmetric quadratic forms*, so $D = k$, σ is trivial, and $\epsilon = +1$. The isometry group is an *orthogonal group*. It is crucial that the anisotropic subspace Q be non-trivial. Let x, y be in the hyperbolic plane so that each of x, y is isotropic and $\langle x, y\rangle = 1$. Fix a non-zero vector $v_o \in Q$. In addition to the two obvious isotropic lines kx and ky, there is the line generated by the isotropic vector

$$2v_o + \langle v_o, v_o\rangle(x - y)$$

Thus, pointedly excepting the case that the quadratic space is a sum of hyperbolic planes, we have the thickness of the building.

It remains to consider the case that D is strictly larger than k. The worst-case scenario is that of a hyperbolic plane (over D). Let x, y be isotropic vectors so that $\langle x, y\rangle = 1$. If $\epsilon = -1$ then the k-subspace $kx + ky$ is a non-degenerate alternating space, so contains at least 3 distinct anisotropic k-one-dimensional subspaces: x, y and something of the form $ax + by$ with neither a nor b zero. It is easy to see that no two of these three vectors are D-multiples of each other either, so we have the desired thickness in case $\epsilon = -1$.

Thus, we are left with proving the thickness in the case that D is strictly larger than k, and $\epsilon = +1$. Again let x, y be a hyperbolic pair as in the previous paragraph. We wish to find at least one non-zero $\alpha \in D$ so that $x + \alpha y$ is isotropic. Written out, this condition is

$$0 = \langle x + \alpha y, x + \alpha y\rangle = \langle x, \alpha y\rangle + \langle \alpha y, x\rangle = \alpha^\sigma + \alpha$$

In the case that D is commutative, since the characteristic is not 2 there is some $\alpha \in D$ so that $\alpha^\sigma = -\alpha$. If D is non-commutative and since $(\alpha\beta)^\sigma = \beta^\sigma\alpha^\sigma$ there must be $\alpha \in D$ so that $\alpha^\sigma \neq \alpha$. Then $\alpha - \alpha^\sigma$ is non-zero and has the desired property. This gives the thickness in this case.

This proves the thickness (although we have not yet quite proven that the *whole* complex is a chamber complex. See the next paragraph).

Now we prove that any two maximal simplices in the whole complex X lie inside one of the subcomplexes $A \in \mathcal{A}$. This, together with the fact (proven above) that each $A \in \mathcal{A}$ is a *chamber* complex, will prove that the whole complex X is a chamber complex. The previous discussion would prove that it is *thick*. So, given two maximal isotropic flags

$$U_1 \subset \ldots \subset U_n$$

$$V_1 \subset \ldots \subset V_n$$

we must find a frame $\mathcal{F}$ so that both flags occur in the subcomplex $A = A_{\mathcal{F}} \in \mathcal{A}$ designated by $\mathcal{F}$.

In contrast to $GL(n)$, where we used a Jordan-Holder theorem, here we use the form $\langle , \rangle$ and induction on the 'index' n.

Thus, we consider first the 'index 1 case', that is, where $V = H \oplus Q$ where Q is anisotropic and H is a hyperbolic plane. Given two isotropic D-one-dimensional subspaces V_1 and U_1, we wish to find two isotropic lines λ^+ and λ^- so that $\lambda^+ + \lambda^-$ is a hyperbolic plane and $V_1 = \lambda^+$ and U_1 is either λ^+ or λ^-. If $V_1 + U_1$ is one-dimensional, then $V_1 = U_1$ and we are done. If $V_1 + U_1$ is two-dimensional, then it cannot be totally isotropic, by invoking Witt's theorem, since a maximal totally isotropic subspace here is just one-dimensional. Thus, by default, because the index is 1, it must be that $V_1 + U_1$ is a hyperbolic plane, and we take $\lambda^+ = V_1$ and $\lambda^- = U_1$.

Now we do the induction step. First, we note that we have chains of subspaces

$$U_1 \subset \ldots \subset U_n \subset U_n^\perp \subset U_{n-1}^\perp \subset \ldots \subset U_1^\perp$$

$$V_1 \subset \ldots \subset V_n \subset V_n^\perp \subset V_{n-1}^\perp \subset \ldots \subset V_1^\perp$$

If $U_1 \subset V_n$, then $V_n \subset U_1^\perp$, and we can consider the space $V_1^\perp / V_1$ with its natural non-degenerate form, and do induction on the index n, to prove that there is a subcomplex $A \in \mathcal{A}$ containing both flags. In particular, let $V_i' = (V_i + U_1)/U_1$ and $U_i' = U_i/U_1$, giving flags of totally isotropic subspaces. (The temporary indexing here does *not* match dimension). Suppose we have found a frame $\bar{\mathcal{F}}$ in the quotient, given by the *images of* isotropic lines $\lambda_i^{\pm 1}$ with $2 \leq i \leq n$, so that all the quotients U_i' and V_i' are sums of (the images of) these lines. Then take $\lambda_1^{+1} = U_1$ and for λ_1^{-1} take any line in V which is orthogonal to all the λ_i^{+1} for $i \geq 2$, and so that $\lambda_1^{+1} + \lambda_1^{-1}$ is a hyperbolic plane. The list of lines $\lambda_i^{\pm 1}$ with $1 \leq i \leq n$ is the desired frame for the apartment containing the two given chambers.

If $U_1 \not\subset V_n$, then let i_o be the smallest index such that there is a line λ in V_{i_o} so that

$$V_1 + \lambda = V_1 \oplus \lambda_1$$

is a hyperbolic plane. Then $(V_1 \oplus \lambda)^\perp$ is a non-degenerate space of smaller dimension, and again we can do induction on dimension to prove that there is a subcomplex $A \in \mathcal{A}$ containing both flags. In more detail: let $V_i' = V_{i-1}$ for $2 \leq i \leq i_o$ and $V_i' = V_i \cap U_1^\perp$ for $i > i_o$, with $U_i' = U_i \cap \lambda^\perp$ for $i \geq 2$. (So the temporary indexing here does *not* match dimension). These are flags of totally isotropic subspaces. Suppose we have found (for $i \geq 2$) (suitably orthogonal) hyperbolic planes

$$\lambda_i^{+1} \oplus \lambda_i^{-1}$$

with $2 \leq i \leq n$ so that all the U_i' and V_i' are sums of the λ_i^{+1}. Then take $\lambda_1^{+1} = U_1$ and $\lambda_1^{-1} = \lambda$. Even more simply than in the case treated in the previous paragraph, we have the desired common apartment as designated by this collection of lines.

The last thing to be done, to prove that X is a thick building, is to show that, if a chamber C and a simplex x both lie in two apartments $A, B \in \mathcal{A}$ then there is a chamber-complex isomorphism $f : B \to A$ fixing both x and C *pointwise*. (Recall that the latter requirement is that f should fix x and C and *any face* of either of them). As in the case of $GL(n)$, we will give f by giving a bijection between the lines in the frames specifying the two apartments. This certainly will give a face-relation preserving bijection. And it is simpler to prove the apparently stronger assertion that, given a chamber C lying in two apartments $A, B \in \mathcal{A}$, there is an isomorphism $f : B \to A$ fixing $A \cap B$ pointwise.

Let $\mathcal{F}$ be the frame given by isotropic lines $\lambda_i^{\pm 1}$ forming (suitably orthogonal) hyperbolic planes $H_i = \lambda_i^{+1} \oplus \lambda_i^{-1}$, and let $\mathcal{G}$ be the frame given by isotropic lines $\mu_i^{\pm 1}$ forming (suitably orthogonal) hyperbolic planes $J_i = \mu_i^{+1} \oplus \mu_i^{-1}$. By relabeling and renumbering if necessary, we may suppose that the common chamber C corresponds to the choices of orderings

$$(H_1, \ldots, H_n)$$

$$(J_1, \ldots, J_n)$$

and lines λ_i^{+1} and μ_i^{+1} for all indices i. Then the i-dimensional totally isotropic subspace occurring as vertex of C is

$$\lambda_1^{+1} + \ldots + \lambda_i^{+1} = \mu_1^{+1} + \ldots + \mu_i^{+1}$$

We attempt to define a map

$$f : B \to A$$

on totally isotropic subspaces (vertices) by

$$f : \lambda_{i_1}^{+1} + \ldots + \lambda_{i_m} \to \mu_{i_1}^{+1} + \ldots + \mu_{i_m}^{+1}$$

for any distinct indices $i_1, \ldots, i_m$. Since, by invocation of the Uniqueness Lemma, there is *at most* one such map, this surely ought to be it.

But we must show that f defined in such manner really is the identity on $A \cap B$. To accomplish this, it suffices to show that it is the identity on all 0-simplices in the intersection. If a 0-simplex x is in the intersection then x is a totally isotropic subspace of V which can be written as a sum of some of the λ_i^{+1} and also can be written as a sum of some of the μ_i^{+1}. What we want to show is that, if

$$x = \lambda_{i_1}^{+1} + \ldots + \lambda_{i_m}^{+1} = \mu_{j_1}^{+1} + \ldots + \mu_{j_m}^{+1}$$

then in fact $i_\ell = j_\ell$ for all ℓ. This would certainly assure that $A \cap B$ is fixed pointwise by f. *This argument is essentially identical to the analogous argument for $GL(n)$*, but we can repeat it here for convenience.

Suppose that x is expressed as above but that it is *not* the case that $i_\nu = j_\nu$ for all ν: let ν be the largest (with $1 \le \nu \le m$) so that $i_\nu \ne j_\nu$. Without loss of generality (by symmetry), suppose that $i_\nu < j_\nu$. By hypothesis, making use of the fact that we have everything renumbered conveniently, we have

$$\lambda_1^{+1} + \lambda_2^{+1} + \ldots + \lambda_{j_\nu-2}^{+1} + \lambda_{j_\nu-1}^{+1} = \mu_1^{+1} + \mu_2^{+1} + \ldots + \mu_{j_\nu-2}^{+1} + \mu_{j_\nu-1}^{+1}$$

Summing this subspace with x, we obtain

$$\lambda_1^{+1} + \lambda_2^{+1} + \ldots + \lambda_{j_\nu-1}^{+1} + \lambda_{i_{\nu+1}}^{+1} + \ldots + \lambda_{i_m}^{+1} =$$

$$= \mu_1^{+1} + \mu_2^{+1} + \ldots + \mu_{j_\nu-1}^{+1} + \mu_{j_\nu}^{+1} + \mu_{j_{\nu+1}}^{+1} + \ldots + \mu_{j_m}^{+1}$$

But the left-hand side has dimension

$$(j_\nu - 1) + (m - \nu) = m + j_\nu - \nu - 1$$

while the right-hand side has dimension

$$(j_\nu - 1) + (m - \nu + 1) = m + j_\nu - \nu$$

This is impossible, so it must have been that $i_\nu = j_\nu$ for all ν. This proves the second axiom for a building.

Thus, we have proven that the complex constructed by taking flags of totally isotropic subspaces of a non-degenerate space is indeed a thick building, with an apartment system provided by frames consisting of unordered $2n$-tuples of lines which can be grouped into pairs which form hyperbolic planes (whose sum is orthogonal).

10.3 The action of the isometry group

In the previous section we constructed a thick building. Incidental to the proof that the apartments are thin chamber complexes, we saw that the Coxeter system is (W, S) with $S = \{s_1, \ldots, s_n\}$, where s_i and s_j commute unless $|i - j| = 1$, and $m(s_i, s_{i+1}) = 3$ for $i < n - 1$ and $m(s_{n-1}, s_n) = 4$. We also saw a model of this W as signed permutation group. Again (2.1), this Coxeter system is of said to be of type C_n. Now we should check that G acts strongly transitively, and preserves types (5.2).

Although we know (4.4) that there is an essentially unique labeling on this building, a tangible labeling is available and is more helpful. This is almost exactly as in the case of $GL(n)$.

We define the *type* of a totally isotropic subspace to be its dimension, and define the *type* of a flag of totally isotropic subspaces to be the list of dimensions of the subspaces. From the definition of the incidence geometry, it is clear that no two distinct vertices of a simplex have the same type. And it is immediate that G preserves this notion of type.

First, we prove transitivity on apartments. Consider two apartments specified by frames

$$\mathcal{F} = \{\lambda_1^{+1}, \lambda_1^{-1}, \ldots, \lambda_n^{+1}\lambda_n^{-1}\}$$
$$\mathcal{G} = \{\mu_1^{+1}\mu_1^{-1}, \ldots, \mu_n^{+1}\mu_n^{-1}\}$$

with $\lambda^{+1} + \lambda_i^{-1}$ (suitably orthogonal) hyperbolic planes, and likewise with $\mu_i^{+1} + \mu_i^{-1}$ (suitably orthogonal) hyperbolic planes. Then there is an isometry $g \in G$ so that

$$g(\lambda_i^{\pm 1}) = \mu_i^{\pm 1}$$

for all choices of sign and for all indices i. Indeed, one merely chooses $x_i \in \lambda_i^{+1}, y_i \in \lambda_i^{-1}$ and then $z_i \in \mu_i^{+1}, w_i \in \mu_i^{-1}$ so that

$$\langle x_i, y_i \rangle = \langle z_i, w_i \rangle$$

By Witt's theorem the isometry g given by $gx_i = z_i$ and $gy_i = w_i$ extends to an isometry of the whole space, so extends to an element of the isometry group. Thus, we have the desired *transitivity on apartments*.

As for $GL(n)$, the fact that images of apartments are again apartments is immediate.

Next, we prove that the stabilizer of a given apartment acts transitively on the chambers within that apartment. The chambers within the apartment A specified by the flag $\mathcal{F}$ above are in bijection with orderings of the hyperbolic planes together with a choice of one of the distinguished lines from each plane. The stabilizer of A certainly includes isometries to yield arbitrary permutations of the hyperbolic planes, and also certainly includes isometries switching the two lines within a given hyperbolic plane. Thus, the collection of configurations corresponding to choice of chamber within a given apartment is acted-upon transitively by the stabilizer of the apartment.

This proves the strong transitivity of G on the building made from flags of totally isotropic subspaces. As remarked just above, the preservation of types is trivial once we realize that dimension of subspace will do.

10.4 The spherical BN-pair in isometry groups

By design, the subgroups B in the BN-pairs arising from the action of G on the thick building of type C_n above really are minimal parabolic subgroups in the geometric algebra sense of (7.4). Thus, once again, facts about parabolic subgroups appear as corollaries to results about buildings and BN-pairs.

We wish to look at some aspects of the situation in coordinates. We consider a D-vectorspace V with form $\langle , \rangle$ of index n, in the sense that a maximal totally isotropic subspace has D-dimension n. Thus, we can write

$$V = Q \oplus (H \oplus \ldots \oplus H)$$

where there are n summands of hyperbolic planes H, and where Q is anisotropic of dimension d.

The standard basis for D^{2n+d} is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots$$

As described earlier in our discussion of classical groups, the **standard form** $\langle , \rangle$ on $V = D^{2n+d}$ with a given anisotropic part Q is given by

$$\langle u, v \rangle = v^\top J_{n,Q} u$$

where

$$J_{n,Q} = \begin{pmatrix} 0 & & & & -1 & & \\ & \ddots & & & & \ddots & \\ & & 0 & & & & -1 \\ & & & Q & & & \\ 1 & & & & 0 & & \\ & \ddots & & & & \ddots & \\ & & 1 & & & & 0 \end{pmatrix}$$

The *standard frame* $\mathcal{F}$ is the collection of lines

$$De_1, De_{1+d+n}, De_2, De_{2+d+n}, De_3, De_{3+d+n}, \ldots , De_n, De_{n+d+n}$$

where we have listed them in the pairs whose sums are hyperbolic planes (whose sum is *orthogonal*).

The *standard maximal isotropic flag* is

$$V_1 = De_1 \subset V_2 = De_1 + De_2 \subset \ldots \subset V_n = De_1 + \ldots + De_n$$

The B in the BN-pair is the stabilizer of this flag.

According to the general prescription, we take $\mathcal{N}$ to be the stabilizer in G of the set of lines in the standard frame $\mathcal{F}$. Thus, in a similar fashion as in the case of $GL(n)$, $\mathcal{N}$ consists of *monomial matrices* in G. The subgroup T here consists of monomial matrices lying in the standard minimal parabolic subgroup. As discussed earlier in our treatment of classical groups and geometric algebra, it is not hard to check that T must be the *standard Levi component* (7.4) of the minimal parabolic, consisting of matrices of the form

$$\begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1_d & & & \\ & & & & t_1^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_n^{-1} \end{pmatrix}$$

By comparison to the case of $GL(n)$, it is not so easy in the present case to identify the Weyl group $\mathcal{N}/T$ in concrete terms. But, as also happened in the case of $GL(n)$, in the proof that the building is indeed a building we were led to consider a model (W,S) of the Weyl group W and generators S wherein W appeared overtly as a *signed permutation group*. This gives a good indication of what to expect for matrix representatives for $W=\mathcal{N}/T$.

For example, the subgroup of W identifiable with plain permutations (with no sign changes) has representatives of the form

$$\begin{pmatrix} \pi & & \\ & 1_d & \\ & & \pi \end{pmatrix} \in G$$

where π is an n-by-n permutation matrix. Note that the inverse of a permutation matrix is its transpose, so the indicated matrices really do lie inside the isometry group.

The change-sign at the i^{th} place has a representative

$$\begin{pmatrix} 1_{i-1} & & & & & & \\ & 0 & & & & -1 & \\ & & 1_{n-i} & & & & \\ & & & 1_d & & & \\ & & & & 1_{i-1} & & \\ & 1 & & & & 0 & \\ & & & & & & 1_{n-i} \end{pmatrix}$$

where 1_m denotes an $m\times m$ identity matrix, and the off-diagonal $-1,+1$ occur at the $(i,i+d+n)^{\text{th}}$ and $(i+d+n,i)^{\text{th}}$ places, respectively.

In a fashion similar to what happened for $GL(n)$, here we used just a little information about the signed permutation group in the proof that our building really was a building. In return, our general results prove (again circuitously) that *the signed permutation group is a Coxeter group*, where we use adjacent transpositions and the change-sign on the n^{th} thing as special set S of generators.

And one may prove, for example, that the *longest elements* in this Coxeter group has representative $J_{n,Q}$ above.

10.5 Analogues for similitude groups

As in (7.2), the similitude group $\tilde{G}$ of the form $\langle,\rangle$ is the slightly larger group

$$\tilde{G}=\{g\in GL_D(V):\langle gu,gv\rangle=\nu(g)\,\langle u,\rangle v\}$$

for some $\nu(g)\in k^\times$. We wish to make the observation that this larger group also acts strongly transitively on the thick building of type C_n for G, and also is label-preserving.

The strong transitivity is immediate from that of G. The preservation of type is likewise clear, if the labeling of totally isotropic subspace *by dimension* is used. Then it is clear that the similitude group preserves the labeling.

While $\tilde{G}$ is slightly larger, and likewise the parabolic subgroups are larger, and likewise the group $\mathcal{N}$ attached to a choice of frame, the Weyl group is naturally identifiable with that of G.

Therefore, for any group intermediate between the isometry and similitude groups of the form $\langle , \rangle$ *the previous construction gives a BN-pair, etc.* Again, this all works for any isometry group except the particular orthogonal group $O(n, n)$, which requires special treatment.

Last, we may consider the slightly smaller **special isometry groups** groups obtained from isometry groups by further imposing the condition that the *determinant be* 1. The issue is whether we still have *strong transitivity*, that is, transitivity on pairs (C, A) where C is a chamber contained in an apartment A. There are several cases in which this is easy to check: For symplectic groups the determinant condition is fulfilled automatically, so the symplectic group itself is alread 'special'. For orthogonal groups in odd dimensions the scalar -1 matrix has determinant -1 yet has trivial action on flags, so from what we've already proven we obtain the strong transitivity. More generally, in a space V with a form $\langle , \rangle$, if V is of *odd dimension* the same remark applies, assuring the strong transitivity.

But if the D-dimension is *even* more careful treatment of individual cases is necessary, depending upon the nature of the underlying field.

11. Spherical Oriflamme Complex

- The oriflamme construction for SO(n,n)
- Verificaition of the building axioms
- The action of SO(n,n)
- The spherical BN-pair in SO(n,n)
- Analogues for GO(n,n)

Now we carry out the **oriflamme** construction of a thick building for *special* orthogonal groups $SO(n, n)$, that is, where in addition to an isometry condition we require determinant one. The more obvious construction discussed above, using flags of isotropic subspaces, which works well for all other isometry groups must be altered in a rather unexpected way to obtain a *thick* building.

In the context of the non-obviousness of the 'correct' construction here, use of the term 'oriflamme' can be explained by a combination of the word's etymology and medieval heraldry. The word comes from the medieval Latin *aurea flamma*, meaning 'golden flame'. In medieval times the abbey of Saint Denis near Paris used such a *golden flame* as its banner. Only by *coincidence*, the golden flame was *branched*. By the time of the Hundred Years' War it had come to be the battle standard of the King of France, and its meaning was taken to be an encouragement to *be courageous and not give up*. Ironically, the Coxeter diagram and the 'shape' of the flags retain the *branchedness* but are no longer *golden* nor are they *flames*.

Still, after having dealt with this unexpected and piquant element, the discussion will strongly resemble that for $GL(n)$ and that for *other* isometry groups.

11.1 The oriflamme construction for SO(n,n)

Here we construct the (spherical) building of type D_n. Instead of literal flags of subspaces as used earlier, we must make a peculiar adjustment, using configurations (of subspaces) called *oriflammes*, defined below. At the end of this section we note the Coxeter data obtained incidentally.

Fix a field k. Let V be a $2n$-dimensional k-vectorspace with a non-degenerate symmetric k-bilinear form $\langle , \rangle$. *The crucial hypothesis is that V is an orthogonal sum of n hyperbolic planes.* This is equivalent to the assumption that every maximal totally isotropic subspace of V has dimension n, exactly half the k-dimension $2n$ of V itself (7.2), (7.3).

Let G be the **special isometry group** of V with the form $\langle , \rangle$:

$$G = \{g \in GL_k V) : \langle gu, gv\rangle = \langle u, v\rangle \ \ \forall u, v \quad \text{and} \quad \det g = 1\}$$

We may often write $SO(n,n)$ for G as an emphatic reminder that we consider only this particular case.

The simplicial complex we will describe is a peculiar variant of the complexes considered earlier. Let Ξ be the collection of non-zero *totally isotropic* k-subspaces of V *of dimension not* $n-1$. We define an *incidence relation* $\sim$ on Ξ by writing $x \sim y$ if *either* $x \subset y$ *or* $y \subset x$ *or* if both x, y are n-dimensional *and* $x \cap y$ is $(n-1)$-dimensional.

The associated *flag complex* X is the simplicial complex with vertex set Ξ and simplices which are *mutually incident* subsets of Ξ. That is, the simplices of X are subsets σ of Ξ so that for all $x, y \in \sigma$ we have $x \sim y$. The *maximal simplices* in X are flags of the form

$$V_1 \subset \ldots \subset V_{n-2} \subset V_{n,1}, V_{n,2}$$

of totally isotropic subspaces V_i of V, where the dimension of V_i is i, the dimension of both $V_{n,1}, V_{n,2}$ is n, and where, pointedly, $V_{n-2} \subset V_{n,1} \cap V_{n,2}$ and the latter intersection has dimension $n-1$.

At the same time, we will continue to have need of the simplicial complex $\tilde{X}$ of the sort used earlier. That is, the vertices in $\tilde{X}$ are non-trivial totally isotropic subspaces, and the incidence relation is $x \sim y$ if and only if $x \subset y$ or $y \subset x$.

Remarks: For quadratic spaces of the special sort considered here, there is a natural two-to-one map

$$\Phi : \text{ chambers in } \tilde{X} \to \text{ maximal simplices in } X$$

That is, ϕ maps *maximal flags of totally isotropic subspaces* (as used for all other isometry groups) to the set of *oriflammes*. Indeed, let

$$V_1 \subset \ldots \subset V_n$$

be a maximal totally isotropic flag of subspaces. As noted earlier in assessing the failure of the earlier approach for *these* quadratic spaces, there are *just two* isotropic lines in the non-degenerate two-dimensional quadratic space $Q = V_{n-1}^{\perp}/V_{n-1}$. (This is true of *any* non-degenerate two-dimensional quadratic space). Let λ_1 be the isotropic line in Q so that $V_n/V_{n-1} = \lambda_1$, and let λ_2 be the other isotropic line. For $i = 1, 2$ put

$$V_{(n,i)} = V_{n-1} \oplus \lambda_i$$

Then

$$V_1 \subset V_2 \subset \ldots \subset V_{n-2} \subset V_{(n,1)} \text{ and } \quad V_{(n,2)}$$

is the associated *oriflamme*.

A **frame** $\mathcal{F}$ in the present setting is an unordered $2n$-tuple of lines (one-dimensional D-subspaces) in V, which admit grouping into unordered pairs $\lambda_i^{+1}, \lambda_i^{-1}$ whose sums $H_i = \lambda_i^{+1} + \lambda_i^{-1}$ are *hyperbolic planes* H_i (in the sense of geometric algebra) in V, so that

$$V = H_1 \oplus \ldots \oplus H_n$$

is an *orthogonal* direct sum of all these hyperbolic planes.

We consider the set $\mathcal{A}$ (the anticipated apartment system) of subcomplexes A of X indexed by frames $\mathcal{F}$ in the following manner: the associated subcomplex $A_{\mathcal{F}}$ (anticipated to be an apartment) consists of all simplices σ with all vertices being totally isotropic subspaces ξ (of dimension not $n-1$) expressible as

$$\xi = \lambda_{i_1}^{\epsilon_1} \oplus \ldots \oplus \lambda_{i_d}^{\epsilon_d}$$

for some unordered d-tuple $\{i_1, \ldots, i_d\}$, where for each i the ϵ_i is ± 1.

Remarks: Note that these frames are the same as those used in treating the complex $\tilde{X}$ in the case of all *other* isometry groups. The two-to-one map Φ on *maximal simplices preserves the subcomplexes specified by frames*, as follows. Let $\tilde{A}_{\mathcal{F}}$ be the subcomplex of $\tilde{X}$ consisting of simplices all of whose vertices are sums of the lines in $\mathcal{F}$. Then for any chamber $\tilde{C}$ in $\tilde{A}_{\mathcal{F}}$, it is immediate that $\Phi(\tilde{C}$ lies in the apartment $A_{\mathcal{F}}$ in X.

11.2 Verification of the building axioms

Keep all the notation of the previous section.

The facets F_i of a maximal simplex

$$C = (V_1 \subset \ldots \subset V_{n-2} \subset V_{n,1}, V_{n,2})$$

are in bijection with the subspaces in the flag, by choice of which to omit. In analogy with prior discussions, we will refer to the i^{th} *facet*, where for $1 \leq i \leq n-2$ this specifies omission of the i^{th} subspace, as usual, and for $i = (n, j)$ with $j \in \{1, 2\}$ this means omission of $V_{n,j}$. Thus, the index i assumes values in the set

$$\{1, 2, 3, \ldots, n-3, n-2, (n,1), (n,2)\}$$

The other maximal simplices in X with facet F_i correspond to flags where only allowed change is at the i^{th} spot.

We note that maximal simplices in an apartment A corresponding to the frame $\mathcal{F} = \{\lambda_i^{\pm 1}\}$ as above are in bijection with choices of orderings of the hyperbolic planes $H_i = \lambda_i^{+1} + \lambda_i^{-1}$ and (further) choice of one of the two distinguished lines from the first $n-1$ of these hyperbolic planes, as follows: to a choice $\lambda_{i_1}^{\epsilon_1}, \ldots, \lambda_{i_n}^{\epsilon_n}$ we associate the totally isotropic subspaces

$$V_j = \lambda_{i_1} \oplus \ldots \oplus \lambda_{i_j}$$

for $1 \leq j \leq n-2$ and

$$V_{n,1} = \lambda_{i_1}^{\epsilon_1} \oplus \ldots \oplus \lambda_{i_{n-1}}^{\epsilon_{n-1}} \oplus \lambda_{i_n}^{\epsilon_n}$$

$$V_{n,2} = \lambda_{i_1}^{\epsilon_1} \oplus \ldots \oplus \lambda_{i_{n-1}}^{\epsilon_{n-1}} \oplus \lambda_{i_n}^{-\epsilon_n}$$

Note that the only difference between $V_{n,1}$ and $V_{n,2}$ is in the choice of $\lambda_{i_n}^{\pm 1}$ as last summand. Then take the flag (in the present sense)

$$C = (V_1 \subset \ldots \subset V_{n-2} \subset V_{n,1}, V_{n,2})$$

This bijection is useful in what follows.

First, we prove that each simplicial complex $A \in \mathcal{A}$ really is a *thin chamber complex*. Fix a frame $\mathcal{F}$ and flag C as just above. For each index i, we must ascertain the possibilities for choices of replacements V_i' for the subspace V_i in the flag, where the index i is among $1, 2, \ldots, n-2, (n,1), (n,2)$. Of course, besides the requisite inclusion relations we require that V_i' is a direct sum of the lines λ_i (in order for it to belong in the apartment A). Obviously the cases $i = (n,1), (n,2)$ require a little special treatment, as does the case $i = n-2$ since it interacts with the $(n,1), (n,2)$.

Take $i < n-2$. On one hand, the requirement $V_i' \subset V_{i+1}$ implies that the direct sum expression for V_i' is obtained by omitting one of the lines from the direct sum expression for V_{i+1}. On the other hand, the requirement $V_{i-1} \subset V_i'$ implies that the expression for V_i' cannot omit any of the lines expressing V_{i-1}. Thus, the only choice involved in specifying V_i is the choice of whether to omit $\lambda_{j_i}^{\epsilon_1}$ or $\lambda_{j_{i+1}}^{\epsilon_{i+1}}$ from the expression

$$V_{i+1} = \lambda_{i_1}^{\epsilon_1} \oplus \ldots \oplus \lambda_{j_i}^{\epsilon_i} \oplus \lambda_{j_{i+1}}^{\epsilon_{i+1}}$$

in the case that $i < n-2$.

If $i = n-2$, then the constraint is that

$$V_{n-3} \subset V_{n-2}' \subset V_(n,1) \cap V_{n,2}$$

In addition to the original V_{n-2}, the only other choice inside the subcomplex A would be to replace $\lambda_{n-2}^{\epsilon_n}$ by $\lambda_{n-1}^{\epsilon_{n-1}}$.

If $i = (n,1)$, then the constraints are that $V_{n,1}'$ be totally isotropic, that $V_{n-2} \subset V_{n,1}' \cap V_{n,2}$ and that the intersection $V_{n,1}' \cap V_{n,2}$ have dimension $n-1$. In addition to the original $V_{n,1}$, the only other choice inside the subcomplex A would be

$$V_{n,1}' = V_{n-2} \oplus \lambda_{n-1}^{-\epsilon_{n-1}} \oplus \lambda_n^{-\epsilon_n}$$

A moment's reflection reveals that, in terms of our indexing, this effect is achieved by simultaneously replacing $\lambda_{n-1}^{\epsilon_{n-1}}$ by $\lambda_n^{-\epsilon_n}$ and replacing $\lambda_n^{\pm\epsilon_n}$ by $\lambda_{n-1}^{\mp\epsilon_{n-1}}$. A similar analysis applies to replacement of $V_{n,2}$, of course.

Let $s_1, \ldots, s_{n-2}, s_{n,1}, s_{n,2}$ be the changes in indexing arising from 'motion' across the respective facets, as just noted. Elementary computations show that $s_i s_{i+1}$ is of order 3 for $i < n-2$, that $s_{n-2}s_{n,j}$ is of order 3 for $j = 1, 2$, and that otherwise these changes commute.

As noted above, choice of chamber in the apartment specified by the frame $\mathcal{F}$ corresponds to a choice of an ordering of the n hyperbolic planes $H_i = \lambda_i^{+1} + \lambda_i^{-1}$, and a further choice of one of the two lines from each of the first $n-1$ of these planes. For $i \leq n-2$, the motion across the i^{th} facet interchanges the i^{th} and $(i+1)^{\text{th}}$ hyperbolic plane. This is no different from earlier computations.

For the last two indices one must be attentive. In particular, one must not attach significance to notation: in fact, choice of one of the last two indices is equivalent to a choice of a line $\lambda_n^{\epsilon_n}$ in the *last* hyperbolic plane H_n in the ordering. The motion across the corresponding facet interchanges the $(n-1)^{\text{th}}$ and n^{th} planes H_{n-1} and H_n, and 'chooses' $\lambda_n^{-\epsilon_n}$ in H_n as distinguished line.

To prove that the subcomplex A is a *chamber* complex, by definition we must find a gallery connecting any two maximal simplices. By the previous discussion, this amounts to showing that any choice of ordering of hyperbolic planes and choice of line from among the first $n-1$ can be obtained from a given one by repeated application of the motion-across-facets changes described above. This is an elementary exercise, comparable to verification that the symmetric group is generated by adjacent transpositions for type A_n.

Remarks: As in the earlier examples, we need only very crude information about the group generated by the motions-across-facets in order to prove the building axioms.

Note that, incidental to the above we did observe that there were always exactly *two* choices for maximal simplices (inside A) with a given facet. Thus, indeed, these *apartments are thin chamber complexes.*

Now we consider the issue of the *thickness* of the whole complex. It is to maintain the thickness that the notion of flag is altered in the present context.

We must show that there are at least 3 possibilities for each subspace occurring in these flags, when we drop the requirement that the subspace occur in the subcomplex corresponding to a frame. For $i < n-2$ we want subspaces V_i' so that

$$V_{i-1} \subset V_i' \subset V_{i+1}$$

This choice is a choice of lines in a two-dimensional vectorspace V_{i+1}/V_{i-1}, allowing us at least 3, as in earlier examples. Also for $i = n-2$ we want V_{n-2}' with

$$V_{n-3} \subset V_{n-2}' \subset V_{n,1} \cap V_{n,2}$$

so we are to choose a line in a two-dimensional space.

The novel issue here is understanding possibilities for replacements for $V_{n,j}$. Since this part of the discussion only considers subspaces of $V_{n-2}^\perp$ which contain V_{n-1}, we may as well look at $V_{n-2}^\perp/V_{n-2}$. Thus, it suffices to consider the case that $n = 2$. To replace $V_{n,1}$ with $V_{n,2}$ given, we must find another two-dimensional totally isotropic subspace $V_{n,1}'$ which intersects $V_{n,2}$ in a one-dimensional subspace. Thus, we choose a line λ inside $V_{n,2}$ and then choose an isotropic line μ in $\lambda^\perp$ but not in $V_{n,2}$. Since $\lambda^\perp/\lambda$ is a hyperbolic plane, the choice of *mu* is just that of an isotropic line in a hyperbolic plane, with one choice excluded, that of $V_{n,2}/\lambda$. But it is elementary that *there are only two isotropic lines in a hyperbolic plane* (in a non-degenerate quadratic space). So for each choice of λ there is exactly one remaining choice of μ. Thus, to count the choices altogether, we count the choices of λ. That is, we count the number of lines in a plane. As earlier, this is at least 3 no matter what the field k may be.

This proves the thickness (although we have not yet quite proven that the *whole* complex is a chamber complex. See the next paragraph).

Now we prove that any two maximal simplices in the whole complex X lie inside one of the subcomplexes $A \in \mathcal{A}$. This, together with the fact (proven above) that each $A \in \mathcal{A}$ is a *chamber* complex, will prove that the whole complex X is a chamber complex. The previous discussion would prove that it is *thick*. So, given two maximal flags we must find a frame $\mathcal{F}$ so that both flags occur in the subcomplex $A = A_{\mathcal{F}} \in \mathcal{A}$ specified by $\mathcal{F}$.

At this point we can exercise a tiny bit of cleverness. Using the two-to-one map from maximal flags of totally isotropic subspaces to oriflammes, we can invoke part of the earlier argument for all *other* quadratic spaces.

That is, given two oriflammes C, D, choose maximal isotropic flags $\tilde{C}, \tilde{D}$ which map to C, D, respectively. It was proven earlier, in discussion of all other isometry groups and their buildings, that there is a frame common $\mathcal{F}$ for $\tilde{C}, \tilde{D}$. (This did *not* depend upon *thickness* of the whole complex). Thus, $\mathcal{F}$ is a common frame for the two given oriflammes, as well. That is, we have proven that for any two maximal simplices (oriflammes) there exists a common apartment, as required by the building axioms.

The last thing to be done, to prove that X is a thick building, is to show that, if a chamber C and a simplex x both lie in two apartments $A, B \in \mathcal{A}$ then there is a chamber-complex isomorphism $f : B \to A$ fixing both x and C *pointwise*. (Recall that the latter requirement is that f should fix x and C and *any face* of either of them). As in the case of $GL(n)$ and general isometry groups, we will give f by giving a bijection between the lines in the frames specifying the two apartments. This certainly will give a face-relation preserving bijection. And it is simpler to prove the apparently stronger assertion that, given a chamber C lying in two apartments $A, B \in \mathcal{A}$, there is an isomorphism $f : B \to A$ fixing $A \cap B$ pointwise.

Let $\mathcal{F}$ be the frame given by isotropic lines $\lambda_i^{\pm 1}$ forming hyperbolic planes $H_i = \lambda_i^{+1} \oplus \lambda_i^{-1}$, and let $\mathcal{G}$ be the frame given by isotropic lines $\mu_i^{\pm 1}$ forming hyperbolic planes $J_i = \mu_i^{+1} \oplus \mu_i^{-1}$. We suppose that the apartments $A_{\mathcal{F}}, A_{\mathcal{G}}$ specified by these frames have a common chamber C. Let C be described by the oriflamme

$$V_1 \subset \ldots \subset V_{n-2} \subset V_{(n,1)} \text{ and } \quad V_{(n,2)}$$

Note that also the totally isotropic subspace

$$V_{n-1} = V_{(n,1)} \cap V_{(n,2)}$$

is expressible as a sum of the lines in these frames.

By relabeling and renumbering if necessary, we may suppose that the common chamber C corresponds to the choices of orderings

$$(H_1, \ldots, H_n)$$

$$(J_1, \ldots, J_n)$$

and lines λ_i^{+1} and μ_i^{+1} for indices $i < n$.

As was done in the treatment of general isometry groups, we attempt to define a map

$$f : B \to A$$

on totally isotropic subspaces (vertices) by

$$f : \lambda_{i_1}^{+1} + \ldots + \lambda_{i_m} \to \mu_{i_1}^{+1} + \ldots + \mu_{i_m}^{+1}$$

for any distinct indices $i_1, \ldots, i_m$. (The fact that we only consider totally isotropic subspaces of dimension *not* $n-1$ is not the main point just now).

But we must show that f defined in such manner really is the identity on the whole intersection $A \cap B$. We will see that the issue here is identical to that treated earlier. Indeed, to show that f is the identity on $A \cap B$, it suffices to show that it is the identity on all 0-simplices in the intersection. If a 0-simplex x is in the intersection then x is a totally isotropic subspace of V which can be written as a sum of some of the λ_i^{+1} and also can be written as a sum of some of the μ_i^{+1}. What we want to show is that, if

$$x = \lambda_{i_1}^{+1} + \ldots + \lambda_{i_m}^{+1} = \mu_{j_1}^{+1} + \ldots + \mu_{j_m}^{+1}$$

then in fact $i_\ell = j_\ell$ for all ℓ. This would certainly assure that $A \cap B$ is fixed pointwise by f.

At this point, the argument used for the complex $\tilde{X}$ and other isometry groups can be repeated verbatim. Thus, we have verified the second axiom for a thick building, completing the oriflamme construction and verification of its properties.

Last, we observe what Coxeter data has been obtained. Let us index reflections in the same manner as subspaces have been indexed above: s_1, s_2, $\ldots$, s_{n-3}, s_{n-2}, $s_{(n,1)}$, $s_{(n,2)}$. Looking back at the discussion of what happens when we reflect through the various facets, by an elementary computation we find that $s_i s_{i+1}$ is of order 3 for $i < n-2$, that $s_{n-2} s_{n,j}$ is of order 3 for $j = 1, 2$, and that otherwise these reflections commute. That is, we have obtained the Coxeter system of type D_n.

11.3 The action of SO(n,n)

We have constructed a thick building X associated to a rather special sort of non-degenerate quadratic space, expressible as a sum of n hyperbolic planes. (Of course, if the underlying field is algebraically closed, then *every even-dimensional* non-degenerate quadratic space is of this type).

Incidental to the proof that the apartments are thin chamber complexes, we saw a fairly concrete picture of the Coxeter system of type D_n. Now we should check that $G = SO(n,n)$ acts strongly transitively, and preserves types.

As noted in the previous two constructions, there is an essentially unique labeling on a thick building (4.4). So any convenient labeling we contrive is as good as any other.

As before, it suffices to label *vertices* in the complex X. Totally isotropic subspaces of dimension $\leq n-2$ we can label simply by dimension, as before. To make sense of the phenomena surrounding the n-dimensional totally isotropic subspaces, we need a little more preparation in the direction of geometric algebra, now *keeping track of determinants.*

Let V be a $2n$-dimensional quadratic space which is an orthogonal direct sum of n hyperbolic planes. Let $G = SO(n,n)$ be the group of isometries g of V with det $g = 1$.

Lemma: Elements of the isometry group of a non-degenerate quadratic form have determinant ± 1.

Proof: In coordinates, we imagine the vector space to consist of column vectors, and the quadratic form to be given by

$$\langle v, v\rangle = v^\top Q v$$

for some symmetric matrix Q. Then the matrix g of a linear automorphism is actually an isometry if and only if $g^\top Q g = Q$. Taking determinants, we obtain

$$(\det\ g)^2 \det\ Q = \det\ Q$$

Since Q is non-degenerate its determinant is non-zero, so det $g = \pm 1$. ♣

Proposition: Let Y be a totally isotropic $(n-1)$-dimensional subspace of V. There are exactly *two* totally isotropic n-dimensional subspaces V_1, V_2 contained in $Y^\perp$.

Proof: The quotient $Q = Y^\perp/Y$ is a non-degenerate two-dimensional quadratic space. In fact, it is a hyperbolic plane, since V was a direct sum of hyperbolic planes. Let x, y be a hyperbolic pair in Q, that is, so that

$$\langle x, x\rangle = 0 = \langle y, y\rangle$$

and

$$\langle x, y\rangle = 1 = \langle y, x\rangle$$

Suppose that $ax + by$ is an isotropic vector. Then

$$0 = \langle ax + by, ax + by\rangle = 2ab$$

Thus, since the characteristic is not 2, we have $ab = 0$. Thus, the *only* isotropic vectors in Q are multiples of x and multiples of y. That is, there are just two isotropic lines in Q.

But isotropic lines in Q are in bijection with n-dimensional totally isotropic subspaces inside $Y^\perp$ and containing Y. ♣

Proposition: Let Y, Z be two $(n-1)$-dimensional totally isotropic subspaces of V, and let $f_o : Y \to Z$ be any vectorspace isomorphism. Then there is $g \in G = SO(n, n)$ so that the restriction of g to Y is f_o.

Proof: Invoking Witt's theorem (7.3), there is an isometry $f : V \to V$ which restricts to the map $f_o : Y \to Z$. Since it lies in an orthogonal group, this f has determinant ± 1.

As just noted (and indeed as source of the necessity of considering the oriflamme complex), there are exactly two isotropic lines λ_1, λ_2 in $Y^\perp/Y$ and exactly two isotropic lines μ_1, μ_2 in $Z^\perp/Z$.

Of course, the isometry f maps $Y^\perp$ to itself and maps $Z^\perp$ to itself. Thus, the induced map sends the λ_i to the μ_j *in some order*.

Choose lines $\tilde{\lambda}_i$ inside $Y^\perp$ which map to λ_i. With such choice, let ϕ be an isometry of V which is the identity on $(\lambda_1 + \lambda_2)^\perp$, which interchanges the two lines λ_1, λ_2, and so that ϕ^2 is the identity. (There are just two such). For example, in suitable coordinates on $\tilde{\lambda}_2 + \tilde{\lambda}_2$ the matrix of one such map ϕ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus, det $\phi = -1$.

Then either f or $f\phi$ has determinant 1, and both restrict to f_o on Y since $Y \subset Y^\perp \subset (\lambda_1 + \lambda_2)^\perp$. ♣

Proposition: Let U be a maximal totally isotropic subspace of V. Given an automorphism $\alpha : U \to U$, there is $h \in SO(n, n)$ which restricts to α on U.

Proof: Let U' be another maximal totally isotropic subspace so that $V = U \oplus U'$. Then the map $x \times y \to \langle x, y\rangle$ on $U \times U'$ identifies U' with the linear dual of U. Thus, there is an *adjoint* $\alpha^\top$ which is a linear automorphism of U' so that for all $x \in U$ and $y \in U'$

$$\langle x, \alpha^\top y\rangle = \langle \alpha x, y\rangle$$

Then $h = \alpha \oplus (\alpha^\top)^{-1}$ is certainly an isometry.

Further, either by choice of coordinates in which to compute or by coordinate-free exterior algebra computations, one finds that the determinant of this h is 1, so actually $h \in SO(n,n)$. ♣

Proposition: Let Y be a totally isotropic $(n-1)$-dimensional subspace of V. Let V_1, V_2 be the two totally isotropic n-dimensional subspaces V_1, V_2 contained in $Y^\perp$. Then these two spaces V_1, V_2 are in *distinct* $SO(n,n)$-orbits.

Proof: Now suppose that for some $g \in SO(n,n)$ we had $gV_1 = V_2$. Invoking the previous result, we may adjust g (staying within $SO(n,n)$) so that g is the identity on Y. Then also $gY^\perp = Y^\perp$.

For a linear automorphism h of V stabilizing the subspaces $Y, Y^\perp$ we have well-defined linear automorphisms h_1, h_2 of the quotients $Y^\perp/Y$ and $V/Y^\perp$ (respectively), and by elementary linear algebra

$$\det\ h = \det(h|_Y) \cdot \det(h_1) \cdot \det(h_2)$$

The *non-degenerate* form $\langle , \rangle$ identifies $V/Y^\perp$ with the linear dual space of Y. Thus, for an isometry g, if $g|_Y$ is the identity on Y, then the (adjoint!) map g_2 induced by g on $V/Y^\perp$ is also the identity. Thus, for such g,

$$\det\ g = \det(\text{ map induced by } g \text{ on } Y^\perp/Y$$

But then we are in the two-dimensional (hyperbolic plane) situation again. Then it is easy to see that isometries *interchanging* the two isotropic lines have determinant -1, while isometries *not* interchanging them have determinant $+1$. ♣

Corollary: The special orthogonal group $G = SO(n,n)$ is *transitive* on the set of unordered pairs $V_{(n,1)}, V_{(n,2)}$ of maximal totally isotropic subspaces whose intersection is $(n-1)$-dimensional. There are exactly *two* G-orbits of *maximal* totally isotropic subspaces.

Proof: Let $V_{(n,1)}, V_{(n,2)}$ and $W_{(n,1)}, W_{(n,2)}$ be two unordered pairs of maximal totally isotropic subspaces intersecting in $(n-1)$-dimensional subspaces Y, Z, respectively. Let $g \in SO(n,n)$ be a map so that $gY = Z$. Again, there are exactly two isotropic lines in $Y^\perp/Y$ (respectively, in $Z^\perp/Z$), so there are exactly two n-dimensional totally isotropic subspaces containing Y (respectively, Z). Thus, the isometry g of the first proposition must map the unordered pair $V_{(n,1)}, V_{(n,2)}$ to the unordered pair $W_{(n,1)}, W_{(n,2)}$. By the previous proposition there is *not* any element of $SO(n,n)$ accomplishing the same mapping but reversing the images. ♣

Thus, we can label n-dimensional totally isotropic subspaces according to which of the two orbits they fall into. There is no canonical way to give primacy to one of these orbits over the other if we have not chosen coordinates on the vectorspace V.

Thus, we have arranged a labeling which is preserved by the action of $G = SO(n,n)$. Repeating, *we label totally isotropic subspaces of dimensions*

$\leq n-2$ by dimension, and label maximal totally isotropic subspaces by the $S)(n,n)$-orbit into which they fall.

Remarks: The artifice of using oriflammes to achieve thickness of the building necessitates shrinking the group from $O(n,n)$ to $SO(n,n)$ to preserve the concommitant labeling. Since labelings are unique up to isomorphism, we are assured that the necessity of restricting our attention to $SO(n,n)$ is *genuine*.

Now transitivity on apartments can be proven. Consider two apartments specified by frames

$$\mathcal{F} = \{\lambda_1^{+1}, \lambda_1^{-1}, \ldots, \lambda_n^{+1}\lambda_n^{-1}\}$$
$$\mathcal{G} = \{\mu_1^{+1}\mu_1^{-1}, \ldots, \mu_n^{+1}\mu_n^{-1}\}$$

with $\lambda^{+1}+\lambda_i^{-1}$ hyperbolic planes, and likewise with $\mu_i^{+1}+\mu_i^{=1}$ hyperbolic planes.

Then there is an isometry $g \in G$ so that

$$g(\lambda_i^{\pm 1}) = \mu_i^{\pm 1}$$

for all choices of sign and at least for indices $i < n$. (We are not obliged to try to say more precisely what happens at $i = n$). Indeed, one merely chooses $x_i \in \lambda_i^{+1}, y_i \in \lambda_i^{-1}$ and then $z_i \in \mu_i^{+1}, w_i \in \mu_i^{-1}$ so that

$$\langle x_i, y_i\rangle = \langle z_i, w_i\rangle$$

Invoking the proposition above, the map given by $gx_i = z_i$ and $gy_i = w_i$ extends to an isometry $g \in SO(n,n)$ of the whole space. This gives the desired *transitivity on apartments*.

Next, we prove that the stabilizer of a given apartment in $G = SO(n,n)$ acts transitively on the chambers within that apartment. The chambers within the apartment A specified by the flag $\mathcal{F}$ above are in bijection with orderings of the hyperbolic planes together with a choice of one of the distinguished lines from each plane *except the last*.

The stabilizer of A certainly includes isometries to yield arbitrary permutations of the hyperbolic planes. However, unlike the case of *orthogonal groups*, the *special* orthogonal group $G = SO(n,n)$ does *not* include an isometry exchanging the two lines inside a hyperbolic plane, since such have determinant -1. But G *does* contain isometries which switch the isotropic lines in the i^{th} hyperbolic plan $(i < n)$ *and* switch the isotropic lines in the *last* hyperbolic plane. Since the lines in the *last* plane are *not ordered*, this achieves the desired effect.

This proves the strong transitivity of $G = SO(n,n)$ on the oriflamme building.

Remarks: Although the failure of the simpler isotropic flag construction for $O(n,n)$ may vaguely hint at something like the oriflamme construction, one ought not pretend that the aptness of th oriflamme construction is obvious *a priori*.

11.4 The spherical BN-pair in SO(n,n)

Since the *oriflammes* appearing in the definition of the building for $SO(n,n)$ are not exactly *flags of totally isotropic subspaces*, it is not quite clear that we have achieved the desired end of having minimal parabolics in $SO(n,n)$ appear as stabilizers of chambers.

That is, it is not quite clear that the resulting BN-pair will have the 'B' being a minimal parabolic. But this is not hard to check, as follows.

If $g \in G$ stabilizes an oriflamme

$$V_1 \subset \ldots \subset V_{n-2} \subset V_{(n,1)} \quad \text{and} \quad V_{(n,2)}$$

then g stabilizes the $(n-1)$-dimensional intersection

$$V_{(n,1)} \cap V_{(n,2)}$$

And since $SO(n,n)$ preserves the notion of *label* appropriate here, g cannot interchange $V_{(n,i)}$.

Thus, the stabilizer of this oriflamme is *contained in* a minimal parabolic. Indeed, the stabilizer of this oriflamme stabilizes *two* maximal flags of totally isotropic subspaces:

$$V_1 \subset \ldots \subset V_{n-2} \subset V_{(n,1)} \cap V_{(n,2)} \subset V_{(n,1)}$$

and

$$V_1 \subset \ldots \subset V_{n-2} \subset V_{(n,1)} \cap V_{(n,2)} \subset V_{(n,2)}$$

On the other hand, if $g \in G = SO(n,n)$ stabilizes a maximal flag of totally isotropic subspaces

$$V_1 \subset \ldots \subset V_n$$

then g stabilizes $V_{n-1}^\perp$. The latter contains exactly two n-dimensional totally isotropic subspaces $V_{(n,1)}, V_{(n,2)}$, one of which is V_n. The action of g cannot interchange them, by the observations of the previous section concerning such situation. Thus, g stabilizes the oriflamme, as desired.

Thus, once again, facts about parabolic subgroups will appear as corollaries to results about buildings and BN-pairs.

Remarks: Another peculiarity of the present situation is that, as is evident from the immediately previous discussion and from the previous section, minimal parabolics (stabilizers of oriflammes) stabilize *two* distinct maximal flags of isotropic subspaces. Thus, attempting to designate minimal parabolics by such flags would be troublesome in any case.

We wish to look at some aspects of the situation in coordinates. We consider a $2n$-dimensional k-vectorspace V with form $\langle , \rangle$ of index n, in the sense that a maximal totally isotropic subspace has k-dimension n. Thus, we can write

$$V = H \oplus \ldots \oplus H$$

where there are n summands of hyperbolic planes H.

The standard basis for k^{2n} is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ldots$$

As described earlier in our discussion of classical groups, the **standard form** $\langle , \rangle$ on $V = k^{2n}$ with *no* anisotropic part is

$$\langle u, v \rangle = v^\top J_n u$$

where

$$J_n = \begin{pmatrix} 0 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & -1 \\ 1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 0 \end{pmatrix}$$

The *standard frame* $\mathcal{F}$ is the collection of lines

$$ke_1, ke_{1+n}, ke_2, ke_{2+n}, ke_3, ke_{3+n}, \ldots, ke_n, ke_{2n}$$

where we have listed them in the pairs whose sums are hyperbolic planes.

The *standard maximal isotropic flag* is

$$V_1 = ke_1 \subset V_2 = ke_1 + ke_2 \subset \ldots \subset V_n = ke_1 + \ldots + ke_n$$

The *standard oriflamme* (much less often mentioned in the classical literature!) is, nevertheless, the obvious thing: letting

$$V_{(n,1)} = V_{n-1} + e_n = V_n$$

and

$$V_{(n,2)} = V_{n-1} + e_{n+1}$$

in this notation the *standard oriflamme* is indeed

$$V_1 \subset V_2 \subset \ldots \subset V_{n-2} \subset V_{(n,1)} \text{ and } \quad V_{(n,2)}$$

The B in the BN-pair is the stabilizer of the flag, and is the stabilizer of the oriflamme, *and* (as observed above) the stabilizer of another flag as well.

According to the general prescription, we take $\mathcal{N}$ to be the stabilizer in G of the set of lines in the standard frame $\mathcal{F}$. Thus, in a similar fashion as in the case of $GL(n)$, $\mathcal{N}$ consists of *monomial matrices* in G. The subgroup T here consists of monomial matrices lying in the standard minimal parabolic subgroup. As discussed earlier in our treatment of classical groups and geometric algebra, it is not hard to check that T must be the *standard Levi component* of the minimal parabolic, consisting of matrices of the form

$$\begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1_d & & & \\ & & & & t_1^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_n^{-1} \end{pmatrix}$$

For example, the subgroup of W identifiable with plain permutations (with no sign changes) has representatives of the form

$$\begin{pmatrix} \pi & & \\ & 1_d & \\ & & \pi \end{pmatrix} \in G$$

where π is an n-by-n permutation matrix. Note that the inverse of a permutation matrix is its transpose, so the indicated matrices really do lie inside the (special) isometry group.

11.5 Analogues for GO(n,n)

The similitude group $GO(n,n)$ of the quadratic form $\langle,\rangle$ is the slightly larger group

$$GO(n,n) = \{g \in GL(V) : \langle gu, gv\rangle = \nu(g)\,\langle u,v\rangle\}$$

for some $\nu(g) \in k^\times$, where k is the underlying field. We wish to make the observation that this larger group also acts strongly transitively on the thick building of type D_n for $SO(n,n)$, although it is *not* label-preserving.

The strong transitivity is immediate from that of $SO(n,n)$. Already we have noted that $O(n,n)$ will not preserve the funny labels on the two sorts of n-dimensional totally isotropic subspaces, because $O(n,n)$ has just *one* orbit on all such, while it was shown that $SO(n,n)$ has *two*, whence the labeling.

While $GO(n,n)$ is slightly larger, and likewise the parabolic subgroups are larger, and likewise the group $\mathcal{N}$ attached to a choice of frame, the Weyl group is naturally identifiable with that of $SO(n,n)$.

Therefore, for any group intermediate between the special isometry group $SO(n,n)$ and the similitude group $GO(n,n)$ the previous construction gives a BN-pair in the generalized sense.

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12. Reflections, Root Systems and Weyl Groups

- Hyperplanes, chambers, walls
- Reflection groups are Coxeter groups
- Root systems and finite reflection groups
- Affine reflection groups, special vertices
- Affine Weyl groups

This section starts anew in development of the idea of *reflection* from another, more literal, viewpoint. This complements the more abstract *simplicial* ideas of the first chapter.

Rather than 'make' Coxeter groups as automorphisms of apartments in thick buildings, we now 'make' them in the guise of 'reflection groups'. We prove that all linear and affine reflection groups are Coxeter groups.

To a great extend the things proven here are independent of our prior work. Indeed, the present considerations are *supplemental* to those developments, providing information of a different sort relevant to the affine and spherical cases.

12.1 Hyperplanes, chambers, walls

Generally, for a subset C of a topological space X, let ∂C be the **boundary** of C inside X. The **closure** of such C inside X is denoted $\bar{C}$.

Let $X = \mathbb{R}^n$, with a positive definite inner product $\langle,\rangle$. Given a finite set $x_o, \ldots, x_m \in X$ and a set of real numbers t_i so that $\sum_i t_i = 1$, the corresponding **affine combination** is

$$x = \sum_i t_i x_i \in X$$

The **affine span** of a set of points in X is the collection of all affine combinations taken from that set. A set of points x_i is **affinely independent** if

$$\sum_i t_i x_i = \sum_i t_i' x_i$$

implies $t_i = t_i'$ for all i. The maximal cardinality of a set of affinely independent points is $n+1$, and any set of affinely independent points can be enlarged to such a set with $n+1$ elements.

The *line through* two distinct points $x, y \in X$ is the set of affine combinations $tx + (1-t)y$. The *closed line segment* $[x, y]$ connecting x, y is the collection of points on the line with $0 \leq t \leq 1$. The *half-open segments* $(x, y], [x, y)$ consist of points with $0 < t \leq 1, 0 \leq t < 1$, respectively.

A real-valued function f on X is an **affine functional** if, for all $t \in \mathbb{R}$ and for all $x, y \in X$ we have

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

Similarly, a map $w : X \to X$ is an **affine map** if

$$w(tx + (1-t)y) = tw(x) + (1-t)w(y)$$

for all $t \in \mathbb{R}$ and for all $x, y \in X$.

An **affine hyperplane** in X is the zero-set of a non-constant affine functional.

Elementary linear algebra assures that there exist n affinely independent points in a hyperplane η.

On the other hand, given a hyperplane η, up to non-zero scalar multiples there is a unique affine functional f whose zero-set is exactly η: Indeed, let $x_1, \ldots, x_n$ be affinely independent points in η and x_o not in η so that $x_o, \ldots, x_n$ affinely span X. If f vanishes on η then $f(x_o)$ determines f, since for an affine combination $y = \sum_i t_i x_i$ we have

$$f(y) = \sum_i t_i f(x_i) = t_o f(x_o)$$

Lemma: Let H be a countable collection of hyperplanes in X, and let λ be a line not contained in any $\eta \in H$. Then

$$\lambda \neq \bigcup_{\eta \in H} (\lambda \cap \eta)$$

Proof: Induction on the dimension n of X. If $n = 1$ then hyperplanes are points, and since $\mathbb{R}$ is uncountable no line can be a countable union of points. For the induction step: let ζ be a hyperplane containing λ (and necessarily distinct from all the $\eta \in H$). Then the collection H' of intersections $\zeta \cap \eta$ is a countable collection of hyperplanes contained in $\zeta \approx \mathbb{R}^{n-1}$, no one of which contains λ. (Here we ignore any empty intersections). ♣

In terms of the inner product, an affine hyperplane η may equivalently be described as a set of the form

$$\{x \in X : \langle x - x_o, e_o \rangle = 0\}$$

(where then $x_o \in \eta$ and e_o is any non-zero vector orthogonal to η).

A set H of affine hyperplanes in X is **locally finite** if, given a compact subset K of X, there are only finitely-many $\eta \in H$ so that $\eta \cap K \neq \emptyset$. The set H is necessarily *countable*. For a *locally finite* collection H of affine hyperplanes the **chambers** cut out by H are defined to be the connected components of the complement of $\bigcup_{\eta \in H} \eta$. Since H is locally finite, the chambers are open convex sets.

An affine hyperplane η **separates** two subsets Y, Z of X if there is an affine functional f with zero-set η so that $f > 0$ on Y and $f < 0$ on Z, or vice-versa. Note that since all line segments $[x, y]$ are compact, since chambers are convex, and since H is locally finite, there are only finitely-many walls separating a given pair of distinct chambers cut out by H.

A hyperplane $\eta \in H$ is said to be a **wall** of a chamber C cut out by H if the affine span of $\eta \cap \partial C$ is η. Two chambers C, C' are said to be **adjacent along the wall** or to have the **common wall** $\eta \in H$ if the affine span of $\eta \cap \partial C \cap \partial C'$ is η.

Let H_C be the set of *walls* of a chamber C cut out by a locally finite set of hyperplanes H.

- Given a point y not in the topological closure of C, there is a wall η of C separating y from C.
- Conversely, for every wall η of C there is a point y not in the topological closure of C so among all walls of C *only* η separates y from C.
- For every hyperplane $\eta \in H$, there is at least one chamber of which η is a wall.

Proof: Consider $y \in X$ not in the topological closure of C. Take $x \in C$. Consider the line segment $[x, y)$ and the intersections $\eta \cap [x, y)$. If all of the intersections $[x, y) \cap \eta$ were empty, by continuity we would have $y \in \partial C$.

For fixed y, the collection of $x \in X$ so that the segment $[x, y)$ meets an intersection $\eta \cap \eta'$, for *distinct* η, η' both in H, is a subset of a countable union of hyperplanes. Thus, by the Lemma, we can move x slightly so that points $[x, y] \cap \eta$ are all distinct (or this intersection is empty).

Since one of these intersections is non-empty, there is a *unique* one of these intersections $z = \eta_o \cap [x, y]$ closest to x. Since

$$H' = \{\eta \cap \eta_o : \eta \in H, \eta \neq \eta_o\}$$

is a locally finite set of hyperplanes in η_o, the complement in η_o of the union of the *other* hyperplanes is *open* in η_o. Thus, for x' sufficiently near x, the intersection $z' = [x', y) \cap \eta_o$ lies on no other $\eta \in H$. Since $[x, y]$ meets η_o in a single point, we can choose points $x_1, \ldots, x_n$ near x so that the points $z_i = [x_i, y] \cap \eta_o$ are affinely independent: Given $z_1, \ldots, z_k$ affinely independent with $k < n$, the affine span S_k of $z_1, \ldots, z_k, y$ is contained in some affine hyperplane ζ_k, so there is x_{k+1} near x not in ζ_k, and then $z_{k+1} \notin \zeta_k$ either, since $y \in \zeta_k$. Thus, $z_1, \ldots, z_n$ affinely span η, and η_o is a wall of C.

On the other hand, given a wall η of C, let $x_1, \ldots, x_n$ be n affinely independent points on $\eta \cap \partial C$ which affinely span η. For any wall ζ of C and affine functional f_ζ which is positive on C, we have $f_\zeta(x_i) \geq 0$. In fact, for at least one of the x_i we have $f_\zeta(x_i) > 0$, or else $\zeta = \eta$. Let

$$z = \sum_i \frac{1}{n} x_i$$

Then $f_\zeta(z) > 0$ for $\zeta \neq \eta$, z lies on ∂C, and still $f_\eta(z) = 0$.

In some small-enough neighborhood of z there is a point z' so that still $f_\zeta(z') > 0$ for $\zeta \neq \eta$, and $f_\eta(z') < 0$. That is, only the wall η separates z' from C.

Now let $\eta \in H$. Since η is not the union of the intersections $\zeta \cap \eta$ for $\eta \neq \zeta \in H$, there are points $z \in \eta$ which lie on no other hyperplane in H. A point x near such z but off η lies in some chamber cut out by H of which η must be a wall, by arguments as just above. ♣

Corollary: If C, D are distinct chambers, then there is a wall of C separating them.

Proof: The chamber C is exactly described by inequalities only involving affine functionals whose zero-sets are walls of C. If $x \in D$ satisfied the same inequalities, then by the results above $x \in C$, contradiction. ♣

Proposition: Given a chamber C cut out by H, and given a wall η of C, there is exactly one *other* chamber D cut out by H which has common wall η with C.

Proof: For each $\xi \in H$ choose an affine functional f_ξ so that f_ξ vanishes on ξ and is positive on C. (There exist such since C is a connected components of the complement of the union of all the hypersurfaces in H).

Take a wall η of C, with affinely independent $z_1, \ldots, z_n$ in $\eta \cap \partial C$. Put $z = (\sum z_i)/n$. As in the previous proof, we find that $f_\xi(z) > 0$ for $\xi \neq \eta$. Then for $z' \in X$ near z all $f_\xi(z') > 0$ with $\xi \neq \eta$ are still *positive*. Thus, the set

$$C' = \{x \in X : f_\xi(x) > 0 \quad \forall \xi \neq \eta \text{ and } \quad f_\eta(x) < 0\}$$

is *non-empty*, so is a chamber cut out by H. We have shown that there is at least one other chamber C' sharing the wall η with C.

On the other hand, for $z_1, \ldots, z_n$ affinely independent points in $\partial C \cap \partial D \cap \eta$, let $z = (\sum z_i)/n$. The previous argument shows that for $\xi \neq \eta$, an affine functional F_ξ which is positive on C (respectively, positive on D) must be positive on z, for $\xi \in H$. Thus, the only possible difference between C and D can be that an affine functional f_η vanishing on η is positive on one and negative on the other. Thus, we have shown that there is *exactly one* other chamber sharing the wall η with the given chamber C. ♣

A **gallery** of **length** n connecting two chambers C, D is defined to be a sequence of chambers $C = C_o, C_1, \ldots, C_n = D$ so that C_i is adjacent to C_{i+1}.

The gallery

$$C_o, C_1, C_2, \ldots, C_n$$

crosses the wall η if η is the common wall between two chambers C_i, C_{i+1} for some index i.

12.2 Reflection groups are Coxeter groups

Here we show, among other things, that reflection groups satisfy the *Deletion Condition* (1.7), so are Coxeter groups. In fact, we derive several useful results which will come into play later in discussion of the *geometric realization* of affine Coxeter complexes and affine buildings.

Let $X = \mathbb{R}^n$ as above, and let H be a *locally finite* collection of affine hyperplanes in X.

The **(orthogonal) reflection** through a hyperplane η is the automorphism $s = s_\eta$ of X described by

$$sx = x - \frac{2\langle x - x_o, e_o\rangle}{\langle e_o, e_o\rangle} e_o$$

where x_o is an arbitrary point on η and e_o is any non-zero vector perpendicular to η. One can check that this definition does not depend upon the choices made.

Let G be the group generated by all orthogonal reflections through hyperplanes in H and *suppose that* H *is stable under* G, that is, that if $\eta \in H$ and $g \in G$ then $g\eta = \{gx : x \in \eta\}$ is also in H. This group G is called a **reflection group**.

Remarks: Having made the assumption that the set of hyperplanes is stable under all reflections through members of H, we can sensibly introduce some further standard terminology: If the hyperplanes in H have non-trivial common intersection, the reflection group generated is a **linear reflection group**. If the hyperplanes in H have trivial common intersection, then the group is called an **affine reflection group** and the chambers are called **alcoves**.

Lemma: For two chambers C, D cut out by H, let $\ell = \ell(C, D)$ be the number of hyperplanes in H which separate them. Then there is a gallery of length ℓ connecting them.

Proof: Induction on the number of walls separating C, D. First, if *no* walls separate the two chambers, then (e.g., by the previous section) C, D are defined by the same collection of inequalities, so must in fact be the same chamber. So suppose that $C \neq D$. Let η be a wall of C separating C, D. Let C' be the chamber obtained by reflecting C through η. Then η does *not* separate C' from D, since we have just crossed η in going from C to C'. And we crossed not other hyperplanes in H in going from C to C'. Thus,

$$\ell(C', D) = \ell(C, D) - 1$$

By induction, C', D are connected by a gallery

$$C' = C_1, C_2, \ldots, C_\ell = D$$

of length $\ell - 1$. Then it is easy to see that C, D are connected by the gallery

$$C = C_o, C' = C_1, C_2, \ldots, C_\ell = D$$

of length ℓ. ♣

Let C be a fixed chamber cut out by H, let S be the set of reflections through the hyperplanes in H which are *walls* of C, and let W be the subgroup of G generated by S.

Recall that a group action of a group G on a set X is *simply-transitive* if the action is transitive and if for all $x \in X$ the equality $gx = x$ implies that $g = 1$.

Recall that the *Deletion Condition* on a group W and a set S of generators for W is that if the *length* of a word $s_1 \ldots s_n$ is *less* than n, then there are indices i, j so that

$$s_1 \ldots s_n = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_n$$

That is, the product is unchanged by deletion of s_i and s_j. The least n so that w has an expression $w = s_1 \ldots s_n$ is the *length* $\ell(w)$ of w with respect to the generators S of W.

We prove the following family of related results all together.

- The group W is transitive on chambers cut out by H, and $G = W$.
- The group G is *simply*-transitive on chambers.
- The length $\ell(w)$ of $w \in W$ is the number $\ell(C, wC)$ of walls separating C from wC. Each wall crossed by a minimal gallery from C to wC is crossed just once, and the collection of walls crossed by such a minimal gallery is exactly the collection of walls separating C from wC.
- The topological closure $\bar{C}$ of C is a *fundamental domain* for the action of W on X, in the sense that
$$X = \bigcup_{w \in W} w\bar{C} = X$$
- The *isotropy subgroup* or *stabilizer*
$$W_x = \{w \in W : wx = x\}$$
in W of x in the topological closure $\bar{C}$ of C is the subgroup of W generated by
$$S_x = \{s \in S : sx = x\}$$
- The pair (W, S) satisfies the *Deletion Condition*, so (W, S) is a Coxeter system.

Proof: Prove transitivity on chambers by induction on the length of a gallery from C to another chamber D. Let η be a wall of D separating C from D, and let D' be the chamber obtained by reflecting D across η. Then $\ell(C, D')$ is one less than $\ell(C, D)$, so there is $w \in W$ so that $wC = D'$. Then $w^{-1}D$ is a chamber adjacent to $w^{-1}D' = C$. Let s be the reflection across the common wall of $w^{-1}D$ and C. Then certainly $sC = sw^{-1}D' = w^{-1}D$. That is, $D = wsC$, as desired. This is the transitivity.

Let η be a wall of a chamber D, and take $w \in W$ so that $wC = D$. Then $w^{-1}\eta$ is a wall of C, and the reflection t through η is simply $t = wsw^{-1}$ where

s is the reflection through $w^{-1}\eta$. Thus, W contains all reflections through walls, so contains G.

Let $w = s_1 s_2 \ldots s_m$ be an expression for w in terms of $s_i \in S$. Then

$$C = C_o,\ C_1 = s_1 C,\ C_2 = s_1 s_2 C,\ C_3 = s_1 s_2 s_3 C,\ \ldots, C_m = wC$$

is a gallery from C to wC. Therefore, it is clear that

$$\ell(w) \leq \ell(C, wC)$$

If $\ell(w) > \ell(C, wC)$ then some wall is crossed at least twice by the gallery.

The hyperplanes crossed by this gallery are described as follows. Let

$$w_i = s_1 \ldots s_i$$

Then $C_i = w_i C$, and

$$C_{i+1} = w_{i+1} C = w_i s_{i+1} C = w_i s_{i+1} w_i^{-1} w_i C = w_i s_{i+1} w_i^{-1} C_i$$

Thus, C_{i+1} is obtained from C_i by reflecting by $w_i s_{i+1} w_i^{-1}$.

The assumption that a wall is crossed twice is the assumption that for some $i < j$

$$w_i s_{i+1} w_i^{-1} = w_j s_{j+1} w_j^{-1}$$

Then, using $i < j$, we have

$$s_{i+1} = (s_{i+1} \ldots s_j) s_{j+1} (s_{i+1} \ldots s_j)^{-1}$$

from which we obtain, upon right-multiplying by $s_{i+1} \ldots s_j$,

$$s_{i+2} \ldots s_j = s_{i+1} \ldots s_{j+1}$$

Then

$$w = s_1 \ldots s_n = s_1 \ldots \hat{s}_{i+1} \ldots \hat{s}_{j+1} \ldots s_n$$

That is, we can remove s_{i+1} and s_{j+1} from the expression for w as a word in elements of S.

But we could have assumed that the original expression was already the shortest possible, that is, was *reduced*. Thus, we conclude that the length of w is equal to the number of walls separating C from wC, and no wall is crossed twice by a minimal gallery from C to wC. On the other hand, if a wall η is not crossed by a gallery from C to wC, then the gallery stays to one side of the hyperplane η, so η does not separate the two chambers.

In particular, $wC = C$ implies that w is of length zero, so is 1. This gives the simple-transitivity.

Every point in X is in the closure of some chamber, so $\bar{C}$ is a fundamental domain.

Certainly the subgroup of W generated by S_x is contained in the isotropy subgroup W_x. On the other hand, given $x, y \in \bar{C}$, suppose that $wx = y$. We must show that $x = y$ and that w is in the subgroup generated by S_x. This is by induction on the length of w with respect to the generators S of W. Let

$w = s_1 \ldots s_m$ be a reduced expression, that is, of minimal length with $m > 0$. Then

$$C = C_o,\ s_1 C,\ s_1 s_2 C, \ldots,\ (s_1, \ldots s_{m-1})C,\ wC$$

is a minimal gallery from C to wC. This gallery crosses the wall η_1 of C fixed by s_1, so since the gallery is minimal C, wC are separated by the wall η_1. Hence, from the definition, the intersection of their closures is contained in η_1. Then

$$wx = y \in \bar{C} \cap w\bar{C} \subset \eta_1$$

Thus, as necessarily $y \in \eta_1$,

$$(s_1 w)x = s_1 y = y$$

By induction on length, $x = y$. Further, since we saw that $y \in \eta_1$, certainly $x = y \in \eta_1$, so s_1 fixes x, and by induction $w' = (s_1 w)$ is in the subgroup of W generated by S_x.

Observe that we showed that if the length of $w = s_1 \ldots s_m$ is less than m then two factors can be deleted from this product: the Deletion Condition (1.7) holds.

♣

12.3 Root systems and finite reflection groups

If the set H of affine hyperplanes is locally finite, and if the hyperplanes in H have a common point, then the total number of hyperplanes in H is *finite* and we can change coordinates on $X \approx \mathbb{R}^n$ so that the common point is 0. Then all hyperplanes are *linear*, and the associated reflections are likewise *linear*. The associated *finite reflection group* is sometimes also called *spherical*.

We can arrive at this situation by a slightly different route, related to our prior discussion (1.4), (1.5) of *roots*, as follows.

Let Φ be a *finite* collection of vectors in a finite-dimensional real vectorspace V equipped with a positive-definite inner product $\langle , \rangle$. For $\alpha \in \Phi$, let s_α be the corresponding **reflection**: for $v \in V$

$$s_\alpha(v) = v - \frac{2\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$$

The set Φ is a (finite) **root system** if

$$s_\alpha \Phi = \Phi$$

for all $\alpha \in \Phi$. Then the group W generated by the reflections s_α for $\alpha \in \Phi$ is evidently a finite linear reflection group, since it certainly is a subgroup of the finite group of permutations of the finite set Φ.

Say that Φ is a (finite) **reduced root system** if also

$$\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$$

for all $\alpha \in \Phi$. Given a root system Φ, we might replace every $\alpha \in \Phi$ by the corresponding *unit* vector $\alpha/\langle\alpha,\alpha\rangle^{1/2}$, obtaining the associated *reduced* root system Φ'. Visibly, this does not alter the group W obtained. Generally, altering the lengths of roots does not affect the group W obtained, but may affect other aspects of the situation.

The set $\check{\Phi}$ of **co-roots** associated to roots Φ is the set of elements

$$\check{\alpha} = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$$

for $\alpha \in \Phi$. It is easy to check that this is again a root system, called the **dual** root system. The associated group W is the same, again, since the collection of hyperplanes associated to $\check{\Phi}$ is the same as that for Φ.

The root system is **crystallographic** if

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$$

for all $\alpha, \beta \in \Phi$. If the root system is crystallographic (and finite), then the group W is called a **Weyl group**, with reference to the generators s_α for $\alpha \in \Phi$ being implicit. In this definition, altering the lengths of roots certainly *does* matter.

In any case, the collection H of linear hyperplanes

$$\eta_\alpha = \{v \in V : \langle v,\alpha\rangle = 0\}$$

for $\alpha \in \Phi$ is a finite collection of linear hyperplanes, stable under the action of W since Φ is and since W leaves the inner product $\langle,\rangle$ invariant. Thus, the previous discussions apply. Again, note that *replacing a root system Φ by its associated reduced root system gives rise to the same collection of hyperplanes, and the same reflection group W.*

One purpose of this section is to study the *'shape' of the chambers* cut out by a *finite* reflection group: we will see that the chambers are *simplicial cones* (defined below). This study is intimately related to the notion of choice of *simple roots* inside the root system.

So fix a (finite) root system Φ and let H be the associated finite collection of hyperplanes. *For the purposes of this section, without loss of generality we suppose that Φ is reduced, and that the roots are of length* 1.

Fix a chamber C cut out by H, let S be the collection of reflections in the walls of C, and let W be the group generated by these reflections. Let $\pm e_\eta$ be the two *unit* vectors orthogonal to $\eta \in H$. Given the choice of chamber C, the **positive roots** Φ^+ are those roots $\alpha \in \Phi$ so that

$$\langle x,\alpha\rangle > 0 \ \ \forall x \in C$$

From the definition of chamber it follows that

$$\Phi = \Phi^+ \sqcup (-\Phi^+)$$

The set Δ of **simple roots** in Φ^+ is defined to be the set of $\alpha \in \Phi^+$ so that α is *not* expressible as a linear combination of *two or more* elements of

Φ^+ with *positive* coefficients. Then, using the finiteness of Φ and induction, every $\gamma \in \Phi^+$ is expressible as

$$\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

where $c_\alpha \geq 0$ for all α.

From this definition it is clear that Δ is *minimal* among the collection of subsets E of Φ^+ so that all elements of Φ^+ are expressible as linear combinations of elements of E with non-negative coefficients: if $\alpha \in \Delta$ could be omitted, then α would be expressible as a linear combination $\sum_\beta c_\beta \beta$ over $\beta \in \Delta - \{\alpha\}$, with c_β all non-negative. By definition of Δ, at most one of the coefficients c_β can be positive. But then we have an expression $\alpha = c_\beta \beta$. But this is impossible. This proves the minimality.

Lemma: A point $x \in X$ lies in the chamber C if and only if for all $\alpha \in \Delta$ we have $\langle x, \alpha \rangle > 0$.

Proof: If $x \in C$, then $\alpha \in \Delta \subset \Phi^+$ gives $\langle x, \alpha \rangle > 0$. On the other hand, if $\langle x, \alpha \rangle > 0$ for all $\alpha \in \Delta$ then $\langle x, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$, since all elements of Φ^+ are non-negative linear combinations of elements of Δ (with *some* strictly positive coefficient present). Then since C is a connected component of the complement of the union of all the hyperplanes $\langle *, \alpha \rangle = 0$, we find that $x \in C$. ♣

Lemma: For distinct $\alpha, \beta \in \Delta$, we have

$$\langle \alpha, \beta \rangle \leq 0$$

Proof: Throughout the proof, keep in mind that $\langle x, \alpha \rangle > 0$ for all $x \in C$ and for all $\alpha \in \Phi^+$. And, for this proof, we may suppose without loss of generality that α, β are *unit* vectors.

Suppose that $\langle \alpha, \beta \rangle > 0$ for a pair $\alpha, \beta \in \Delta$. Let s be the reflection in the hyperplane orthogonal to α. Then $s\beta$ is again in Φ, since H was stable under all these reflections.

Suppose $s\beta \in \Phi^+$. Write $s\beta = \sum_\alpha c_\alpha \alpha$ with non-negative coefficients, and $\alpha \in \Delta$. If $c_\beta < 1$, then we rearrange to obtain

$$(1 - c_\beta)\beta = 2\langle \beta, \alpha \rangle + \sum_{\gamma \neq \beta} c_\gamma \gamma$$

That is, β is expressible as a non-negative linear combination of elements from $\Delta - \{\beta\}$, contradicting the minimality of Δ. If $c_\beta \geq 1$, then we rearrange to obtain

$$0 = (c_\beta - 1)\beta + 2\langle \beta, \alpha \rangle + \sum_{\gamma \neq \alpha, \beta} c_\gamma \gamma$$

Taking inner product with any $x \in C$ gives $0 < 0$, contradiction.

Suppose that $-s\beta \in \Phi^+$. Write $-s\beta = \sum_\alpha c_\alpha \alpha$ with non-negative coefficients, summed over $\alpha \in \Delta$. If $c_\alpha - 2\langle\beta,\alpha\rangle \geq 0$, then we rearrange to obtain

$$0 = \beta + (c_\alpha - 2\langle\beta,\alpha\rangle)\alpha + \sum_{\gamma\neq\alpha} c_\gamma \gamma$$

Taking inner products with any $x \in C$ gives $0 < 0$, which is impossible. If, on other hand, $c_\alpha - 2\langle\beta,\alpha\rangle < 0$, then we rearrange to obtain

$$(2\langle\beta,\alpha\rangle - c_\alpha)\alpha = \beta + \sum_{\gamma\neq\alpha} c_\gamma \gamma$$

The coefficient of α is positive, so α is expressed as a non-negative linear combination of elements of $\Delta - \{\alpha\}$, contradicting the minimality of Δ.

This excludes all the possibilities, so the assumption $\langle\alpha,\beta\rangle > 0$ yields a contradiction. ♣

Corollary: The simple roots are linearly independent. The collection of hyperplanes orthogonal to the simple roots is exactly the collection of walls of the chamber C. The chamber C has at most $n = \dim X$ walls.

Proof: If the simple roots were *not* linearly independent, then we could write

$$v = \sum_{\alpha\in I} a_\alpha \alpha = \sum_{\beta\in J} b_\beta \beta$$

for some $v \in X$, where I, J were disjoint subsets of Δ, with all a_α, b_β strictly positive. Then

$$0 \leq \langle v,v\rangle = \sum_{\alpha,\beta} a_\alpha b_\beta \langle\alpha,\beta\rangle \leq 0$$

From this, $v = 0$. But then for $x \in C$ we have

$$0 = \langle x, 0\rangle = \langle x, \sum a_\alpha \alpha\rangle = \sum a_\alpha \langle x,\alpha\rangle$$

Since $\langle x,\alpha\rangle > 0$, this would force $I = \emptyset$. Similarly, $J = \emptyset$.

Thus, there could have been no non-trivial relation, so the simple roots are linearly independent, so there are at most $n = \dim X$ of them.

Since the simple roots are linearly independent, and since C is the set of x so that $\langle x,\alpha\rangle > 0$ for all $\alpha \in \Delta$, the linear hyperplanes

$$\eta_\alpha = \{x \in X : \langle x,\alpha\rangle = 0\}$$

perpendicular to $\alpha \in \Delta$ are exactly the walls of C. Indeed, by the linear independence, given $\alpha \in \Delta$, there is $v \in X$ so that $\langle v,\alpha\rangle = 1$ and $\langle v,\beta\rangle = 0$ for $\alpha \neq \beta \in \Delta$. Then, given $x \in C$, for suitable real numbers t the point $y = x + tv$ yields

$$\langle y,\alpha\rangle = \langle x,\alpha\rangle + t < 0$$
$$\langle y,\beta\rangle = \langle x,\beta\rangle > 0$$

That is, η_α is the only hyperplane separating C from y. By the elementary results on walls of chambers, this proves that η_α is a wall of C. ♣

Now let

$$H_C = \{\eta_1, \ldots, \eta_m\}$$

be the walls of C. Let α_i be a root orthogonal to η_i, and from the two possibilities for α_i choose the one so that $\langle x, \alpha_i\rangle > 0$ for $x \in C$. That is, from above, the α_i are the *simple roots*.

The group W is called **essential** if W has no non-zero fixed vectors on X, that is, if $wx = x$ for all $w \in W$ for $x \in X$ implies $x = 0$.

A **simplicial cone** in X is a set of the form

$$\{\sum_{1\leq i\leq n} t_i x_i : \forall t_i \geq 0\}$$

where $e_1, \ldots, e_n$ is a fixed $\mathbb{R}$-basis for X.

Corollary: Suppose that W is essential. Then the chamber C is a *simplicial cone.*

Proof: Since W is essential, it must be that

$$\bigcap_{\eta\in H} \eta = \{0\}$$

Since (by the previous section) *all* the reflections in $\eta \in H$ are in W, in fact it must be that

$$\bigcap_{\eta\in H_C} \eta = \{0\}$$

Therefore, $m \geq n$.

On the other hand, we just showed that the number of walls is $\leq n$ and the e_i are linearly independent. Thus, we can find x_i so that $\langle \alpha_j, x_i\rangle = 0$ for $j \neq i$ and $\langle \alpha_i, x_i\rangle = 1$. Then the chamber C can indeed be described as the set of elements in X of the form $\sum t_i x_i$ with all $t_i > 0$. That is, C is a simplicial cone. ♣

Remarks: In general, if W is *not* necessarily essential, then we can write $X = X_o \oplus X'$ where W acts trivially on X_o, stabilizes X_1, and the action of W on X' *is* essential. Then the chambers are cartesian products of the form

$$X_o \times \text{ simplicial cone in } X'$$

Corollary: The reflections s_α for $\alpha \in \Delta$ generate W.

Proof: The reflections attached to simple roots are the reflections in the walls of the chosen chamber, which do generate the whole group W, by the general results on reflection groups. ♣

Corollary: With Φ reduced, given a root γ, there is $w \in W$ so that $w\gamma \in \Delta$.

Proof: By replacing γ by $-\gamma$ if necessary, we may suppose that γ is a positive root. Since W is finite, there is indeed an element of W which sends all positive roots to negative: this is the *longest* element of W.

Let

$$\delta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

be the element of $W\gamma \cap \Phi^+$ with the smallest *height* $\sum c_\alpha$. Since everything is finite and since at least γ itself lies in this set, we are assured that such element exists. Then

$$0 < \langle \delta, \delta \rangle = \sum_\alpha c_\alpha \langle \alpha, \delta \rangle$$

so certainly there is a simple root α so that $\langle \alpha, \delta \rangle > 0$. If already $\delta \in \Delta$ then we are done.

Suppose that δ is *not* simple. Recall, from our elementary discussion of Coxeter groups, that for $\alpha \in \Delta$ the reflection s_α sends the root α to $-\alpha$ and merely permutes the other positive roots. (It is here that we make use of the *reduced-ness* of the root system).

Thus, $s_\alpha \delta$ must still be *positive*. Since

$$s_\alpha \delta = \delta - \frac{2\langle \delta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

the height of $s_\alpha \delta$ is no larger than that of δ. We contradict the definition of δ unless $\langle \delta, \alpha \rangle = 0$. But this must hold for every $\alpha \in \Delta$, so δ is fixed by every s_α. Since the latter reflections generate W, δ is fixed by W, contradicting the fact that δ is certainly not fixed by its own associated reflection s_δ. ♣

12.4 Affine reflection groups, special vertices

Let H be a *locally finite* set of affine hyperplanes in $X \approx \mathbb{R}^n$. In contrast to the previous section, we now suppose that there is *no* point common to all the hyperplanes. Under some additional hypotheses (below), we will show that chambers cut out by H are *simplices*.

We suppose that H is stable under reflections through $\eta \in H$. Fix a chamber C cut out by H, let S be the collection of reflections in the walls of C, and let W be the group generated by these reflections. (We have shown that (W, S) is a Coxeter system).

Suppose also that (W, S) is *indecomposable* in the sense that the Coxeter diagram is *connected*, that is, S cannot be partitioned into two sets S_1, S_2 so that $s_1 s_2 = s_2 s_1$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Let

$$H_C = \{\eta_o, \ldots, \eta_m\}$$

be the walls of C. Let e_i be a unit vector orthogonal to η_i, let y_i be a point in η_i, and from the two possibilities for e_i choose the one so that

$$\langle x - y_i, e_i \rangle > 0$$

for $x \in C$.

Let H_C be the set of walls of C. For $\eta \in H_C$, the **inward-pointing unit normal vector** $e = e_\eta$ is the unit vector orthogonal to η so that for $x \in C$ and $x_\eta \in \eta$ we have

$$\langle x - x_\eta, e_\eta \rangle > 0$$

Lemma: For distinct walls $\eta \neq \zeta$ of C with inward-pointing unit normal vectors e, f (respectively), we have

$$\langle e, f \rangle \leq 0$$

Proof: First, we claim that if e, f are parallel, then $\zeta = -\eta$, so that $\langle \eta, \zeta \rangle = -1$. If $\zeta \neq -\eta$, then necessarily $\zeta = \eta$. But then it is easy to see that only one of the two hyperplanes could be a wall of C, contradiction. Thus, $\zeta = -\eta$ as claimed.

Now consider e, f not parallel. Then η and ζ have a common point of intersection, which we may suppose to be 0, by changing coordinates. The subgroup W' of W generated just by the *linear* reflections in η, ζ has a unique chamber C' containing C, and η, ζ are still walls of C', from the definition of 'wall'. Let H' be the collection of images of η, ζ under W'. Since H was locally finite, certainly H' is locally finite. Further, H' consists of hyperplanes through 0. The results of the previous section are now applicable to W' and H'. In particular, we have

$$\langle \eta, \zeta \rangle \leq 0$$

as desired. ♣

Corollary: There are only finitely-many parallelism classes of hyperplanes in H.

Proof: If there were infinitely-many hyperplanes in H no two of which were parallel, then the inward-pointing unit normal vectors would have an *accumulation point* on the (compact!) unit sphere in X. In particular, the cosines $\langle e_\eta, e_\zeta \rangle$ of the angles would get arbitrarily close to 1 for distinct $\eta, \zeta \in H$. But the lemma shows that this is impossible. ♣

For $w \in W$, since w is an affine map, we can write

$$wx = \bar{w}x + T_w$$

where the **linear part** $\bar{w}$ of w is a linear map $X \to X$ and where $T_w \in X$ is the **translation part** of w. Of course, this decomposition depends upon what point we call 0, so a change of coordinates moving 0 would change this decomposition.

For $w \in W$, (implicitly depending on choice of 0) let $w \to \bar{w}$ be the map from W to the group $\overline{W}$ of *linear parts*. One can readily check that this map is a *group homomorphism*. The kernel W_1 of the map $W \to \overline{W}$ is the subgroup of **translations** in W. Indeed, for $w \in W_1$ and $x \in X$ we have $wx = x + T_w$ for some $T_w \in X$ depending only upon w, not upon x.

Proposition: The group $\overline{W}$ is a finite (linear) reflection group. There is at least one point $x \in X$ so that the stabilizer W_x maps *isomorphically* to $\overline{W}$. The *translation parts* T_w of $w \in W$ lie in W.

Remarks: A point x so that $W_x \to \overline{W}$ is an isomorphism is called **special** or **good**. The proof below shows that *always* $W_x \to \overline{W}$ is *injective*, so the real issue is surjectivity. And we paraphrase the proposition as

Corollary: There exist special vertices in an affine Coxeter complex. ♣

Proof: For each $\eta \in H$ let $\bar{\eta}$ be a hyperplane parallel to η but through 0. We just showed that the family

$$\bar{H} = \{\bar{\eta} : \eta \in H\}$$

is finite; now we show that it is stable under the reflections through elements of $\bar{H}$. Given $\eta, \zeta \in H$, let $\bar{s}$ be the reflection through $\bar{\eta}$. Let t be the reflection through ζ, and $\bar{t}$ the reflection through $\bar{\zeta}$. The hyperplane $\bar{\zeta}$ is the fixed-point set of $\bar{t}$. The image of $\bar{\zeta}$ under $\bar{s}$ is the fixed-point set of

$$\bar{s}\bar{t}\bar{s}^{-1} = \overline{sts^{-1}}$$

since $w \to \bar{w}$ is a group homomorphism. Since sts^{-1} is the reflection through $s\zeta$, its fixed-point set is a hyperplane in H, so its image in $\bar{H}$ is indeed the fixed point set of $\bar{s}\bar{t}\bar{s}^{-1} = \overline{sts^{-1}}$. Thus, $\overline{W}$ is a finite linear reflection group, as claimed.

Let $\eta_1, \ldots, \eta_m$ be distinct elements of H so that the *linear* hyperplanes $\bar{\eta}_i$ are the distinct elements of $\bar{H}$. Since the latter all pass through 0, there must be *some* point x common to all of $\eta_1, \ldots, \eta_m$. Certainly the reflections s_i through the η_i stabilize x and have images in $\overline{W}$ which generate $\overline{W}$. Thus, $W_x \to \overline{W}$ is *onto*. On the other hand, if $w \in W_x$ has $\bar{w} = 1$, then necessarily w is a translation fixing x, which is impossible unless $w = 1$.

To see that all translation parts T_w of $w \in W$ lie in W, let x be a special point and take $w_x \in W_x$ so that $\bar{w}_x = \bar{w}$. Then $T_w = w_x^{-1}w$. ♣

Corollary: For any *special vertex* x, the group W is the semidirect product of the translation subgroup W_1 and the group $\overline{W}$ of *linear parts*:

$$W \approx W_1 \times\!\lhd \overline{W}$$

Proof: The only thing to check is that W_1 is a normal subgroup of W, which is easy, since the group of translations is a normal subgroup of the group of *all* affine automorphisms of X. ♣

Now we assume that *the collection of inward-pointing unit normal vectors to the walls of a chamber C span the vectorspace X*. This assumption is equivalent to the assumption that $\overline{W}$ is *essential*, that is, has no non-zero fixed-vectors in X. For present purposes, an *n-simplex* in X with one vertex at the origin is described as follows: let $f_1, \ldots, f_n$ be a basis for the linear dual of X, and for a positive constant c define

$$\sigma = \{x \in X : f_i(x) > 0 \quad \forall i, \quad \text{and} \sum_i f_i(x) < c\}$$

This is the sort of n-simplex we will see.

Proposition: Suppose that $\overline{W}$ is *essential* and that W is *indecomposable*. Then a chamber cut out by H is an *n-simplex*, where $X \approx \mathbb{R}^n$.

Proof: Let $\eta_o, \ldots, \eta_m$ be the walls of C. Since $\overline{W}$ is essential, the unit normal vectors $e_o, \ldots, e_m$ to the walls must span X. Further, since we are assuming that the walls have no common intersection, $m \geq n$. Therefore, there is a non-trivial linear relation $\sum_i c_i e_i = 0$ among these vectors. Let I be the set of indices i so that $c_i > 0$ and let J be the set of indices j so that $c_j < 0$. Then we can rewrite the relation as

$$\sum_{i \in I} c_i e_i = \sum_{j \in J} (-c_j) e_j$$

Let $v = \sum_{i \in I} c_i e_i$. Then

$$0 \leq \langle v, v \rangle = \langle \sum_{i \in I} c_i e_i, \sum_{j \in J} (-c_j) e_j \rangle =$$

$$= \sum_{i \in I, j \in J} c_i (-c_j) \langle e_i, e_j \rangle \leq 0$$

since the inner products $\langle e_i, e_j \rangle$ are non-positive, from above. If neither I nor J is empty, the indecomposability of W implies that some one of these inner products is non-zero, yielding the impossible conclusion $0 > 0$. Thus, one of I, J must be empty.

Taking $\emptyset \neq I$, we have

$$0 = \sum_{i \in I} c_i e_i$$

If $I \neq \{0, 1, \ldots, m\}$, then there is an index $j \notin I$, and

$$0 = \langle e_j, 0 \rangle = \langle e_j, \sum_{i \in I} c_i e_i \rangle =$$

$$= \sum_{i \in I} c_i \langle e_j, e_i \rangle \leq 0$$

Again by the indecomposability, some one of these inner products is negative, and we again obtain the impossible $0 > 0$. Thus, it must have been that I was the whole set of indices $\{0, \ldots, m\}$.

Note that we have shown that the *only* possible non-trivial relation among the e_i must involve *all* of them. Therefore, it must be that $m = n$ exactly, so that there are exactly $n+1$ walls to C.

Further, we have the relation

$$\sum_{0\leq i\leq n} c_i e_i = 0$$

with some c_i all *positive* (without loss of generality). Then we can suppose (by changing coordinates) that $\eta_1, \ldots, \eta_n$ have common intersection $\{0\}$, and that η_o does *not* pass through 0.

The chamber C is defined by inequalities $\langle x, e_i\rangle > 0$ for $1 \leq i \leq n$ and $\langle x - x_o, e_o\rangle > 0$ for some $x_o \in \eta_o$. The latter can be rearranged to

$$\langle x, \sum c_o^{-1} e_i\rangle < -\langle x_o, e_o\rangle$$

Since we know that $C \neq \emptyset$, necessarily the constant $c = -\langle x_o, e_o\rangle$ is *positive*. Since $c_o^{-1}c_i > 0$, we can rewrite each $\langle x, e_i\rangle > 0$ as $\langle x, c_o^{-1}c_i e_i\rangle > 0$. Thus, taking

$$f_i(x) = \langle x, c_o^{-1} c_i e_i\rangle$$

(for $i > 0$) the defining relations for C become: $f_i > 0$ for $i > 0$ and

$$\sum_{i>0} f_i(x) < c$$

Again emphasizing that the linear functionals f_i are a basis for the linear dual of X, it is clear that C is a simplex. ♣

Proposition: Suppose that $\overline{W}$ is *essential* and that W is *indecomposable*. Then the normal subgroup W_1 of translations in W is a *discrete* subgroup of the group $T \approx \mathbb{R}^n$ of all translations of $X \approx \mathbb{R}^n$. Further, the quotient X/W_1 of X by W_1 is *compact*.

Proof: Now using the vectorspace structure of $X \approx \mathbb{R}^n$, we identify W_1 with an additive subgroup of X by $w \to T_w$.

The images wC of the chamber C under $w \in W_1$ are disjoint. Thus, for fixed $x_o \in C$ the set

$$U = C - x_o = \{v - x_o : v \in C\}$$

is a neighborhood of 0 so that $W_1 \cap U = \{0\}$. Thus, W_1 is discrete.

On the other hand, $\bigcup w\overline{C} = X$. Let

$$Y = \bigcup_{\bar{w}\in\overline{W}} \bar{w}\overline{C}$$

Since $\overline{W}$ is finite (from above), and since C is a simplex, the topological closure $\overline{C}$ of C is compact, and Y is compact. It is clear that

$$\bigcup_{w_1\in W_1} w_1 Y = X$$

Therefore, X/W_1 is compact. ♣

12.5 Affine Weyl groups

From *finite crystallographic* root systems we construct affine reflection groups W_a.

The infinite Coxeter groups W_a so constructed are called **affine Weyl groups** and the chambers cut out by the reflecting hyperplanes are sometimes called **alcoves**.

Let Φ be a finite crystallographic root system, and let W be the corresponding finite linear reflection group, which we have seen is necessarily a Coxeter group. More precisely, if S is the set of reflections in the walls of a chamber, then (W, S) is a Coxeter system.

Since Φ is assumed to be crystallographic, we have

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$$

for all roots $\alpha, \beta \in \Phi$. Again, this notion *is* sensitive to changes in length, so we should *not* normalize roots to have length 1. Again, the *coroot* $\check{\alpha}$ associated to α is

$$\check{\alpha} = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$$

For Φ crystallographic, we have

$$\frac{4\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle} \in \mathbb{Z}$$

As in the usual description of Coxeter data, let $m(s_\alpha, s_\beta)$ be the least positive integer m so that

$$(s_\alpha s_\beta)^m = 1$$

Let e_α be the unit vector $\alpha/\langle\alpha,\alpha\rangle^{1/2}$. From discussion of reflection groups, we know that

$$-\cos(\pi/m) = \langle e_\alpha, e_\beta\rangle \leq 0$$

From these observations, we see that the choices for $m = m(s_\alpha, s_\beta)$ (with $\alpha \neq \beta$ are limited: we can have only

$$\langle e_\alpha, e_\beta\rangle = -1, -\sqrt{3}/2, -\sqrt{2}/2, -1/2, 0$$

with corresponding

$$m = \infty, 6, 4, 3, 2$$

If the group W is assumed *finite*, then ∞ cannot occur, since otherwise there would be an infinite dihedral group occurring as a subgroup.

We further suppose that Φ is *reduced*, so that $\pm\alpha$ are the only multiples of a given root α which are again roots.

We may suppose without loss of generality that the action of W is *essential*. Here this amounts to requiring that Φ span the ambient vectorspace V.

Fix a chamber C for Φ, with corresponding choice Δ of simple roots and choice S of generators for W: this choice is that $\alpha \in \Phi$ is simple if and only if the hyperplane η_α fixed by the reflection s_α is a *wall* of C.

Let Λ be the collection of all integer linear combinations of simple roots. The hypothesis that Φ is crystallographic assures that Λ is stable under all the reflections s_α for $\alpha \in \Phi$. In our discussion of finite reflection groups we showed that the simple roots are linearly independent. Our assumption that W is essential assures that Δ *spans* the vectorspace. This Λ is the **root lattice** attached to Φ, a terminology which is justified by the corollary below.

Lemma: If Φ is crystallographic and *reduced*, then all roots are *integer* linear combinations of simple roots.

Proof: From our discussion of finite reflection groups just above, given a root γ there is $w \in W$ so that $w\gamma \in \Delta$. Also, the reflections s_α attached to Δ generate W. If γ is an integer linear combination of $\alpha \in \Delta$, then for $\beta \in \Delta$ we see that

$$s_\beta \gamma = \gamma - \frac{2\langle \gamma, \beta\rangle}{\langle \beta, \beta\rangle}\beta$$

still has that property, because of the crystallographic hypothesis. Thus, $\Phi = W\Delta$ consists of integer linear combinations of simple roots. ♣

Recall that a $\mathbb{Z}$-*lattice* in a real vectorspace $V \approx \mathbb{R}^n$ is a $\mathbb{Z}$-submodule in V with n generators which spans V. Equivalently, a $\mathbb{Z}$-submodule of V is a $\mathbb{Z}$-lattice if the natural map

$$V \otimes_{\mathbb{Z}} \mathbb{R} \to V$$

given by

$$v \otimes r \to rv$$

is an *isomorphism*.

Corollary: The root lattice Λ is a $\mathbb{Z}$-lattice in V, containing the set of roots Φ, and is stable under the action of W. Similarly, the coroot lattice $\Lambda(\check{\Phi})$, consisting of $\mathbb{Z}$-linear combinations of coroots, is a $\mathbb{Z}$-lattice in V and is stable under the action of W.

Proof: In the discussion of finite reflection groups we saw that the simple roots are linearly independent. The assumption that W is essential implies that they span V. Thus, Λ is a Z-lattice. The previous lemma gives $\Phi \subset \Lambda$, and the definition of 'crystallographic' gives the stability under W. The argument is similar for the coroot lattice. ♣

With fixed crystallographic (and essential) finite reflection group W, define a set H of affine hyperplanes

$$\eta_{\alpha,k} = \{v \in V : \langle v, \alpha\rangle = k\}$$

for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Let W_a be the group of affine automorphisms generated by the affine reflections

$$s_{\alpha,k}(v) = v - (\langle v, \alpha\rangle - k)\check{\alpha}$$

This group W_a constructed from the (reduced) finite crystallographic root system Φ is an **affine Weyl group**. Concommitantly, we might say that W is a **spherical** Weyl group when the root system is crystallographic.

For $\lambda \in V$ we have the **translation**

$$\tau_\lambda(v) = v + \lambda$$

Via $\lambda \to \tau_\lambda$ we may identify V with a subgroup of affine automorphisms of V.

Proposition: The collection H of affine hyperplanes $\eta_{\alpha,k}$ is locally finite and is stable under W_a. The affine Weyl group W_a is the semi-direct product

$$W_a = W \lhd \times \Lambda(\check{\Phi})$$

of the group W and the coroot lattice $\Lambda(\check{\Phi})$. The group generated by reflections in the hyperplanes in H is just W_a.

Proof: Certainly W lies inside W_a. Note that

$$\tau_{\check{\alpha}} = s_{\alpha,1}s_\alpha = s_{\alpha,1}s_{\alpha,0}$$

so the group of translations coming from $\Lambda(\check{\Phi})$ also lies inside W_a.

Since also

$$s_{\alpha,k} = \tau_{k\check{\alpha}}s_\alpha$$

we see that the generators $s_{\alpha,k}$ for W_a lie in the group generated by W and $\Lambda(\check{\Phi})$.

It is easy to check that W normalizes the translation group given by $\Lambda(\check{\Phi})$. Thus, W_a is the indicated semi-direct product.

The W-invariance of the inner product and W-stability of the roots Φ immediately yield the W-stability of H. Likewise, if $\langle v, \alpha\rangle = k$ and $\lambda = \check{\beta}$ for $\beta \in \Phi$ is in $\Lambda(\check{\Phi})$, then

$$\langle v + \lambda, \alpha\rangle = k + \langle\check{\beta}, \alpha\rangle$$

and $\langle\check{\beta}, \alpha\rangle$ is integral, by the crystallographic assumption. Thus, the collection of hyperplanes is $\Lambda(\check{\Phi})$-stable.

Since the group W_a is the indicated semi-direct product, and since the *finite* set H_o of *linear* hyperplanes $\eta_{\alpha,0}$ is W-stable, it follows that H is the collection of translates of H_o by the *discrete* translation group $\Lambda(\check{\Phi})$.

Suppose there were infinitely-many hyperplanes $\eta_{\alpha,k}$ within distance $\epsilon > 0$ of a point $x \in V$. Let $y = y_{\alpha,k}$ be a point on $\eta_{\alpha,k}$ within distance ϵ of x:

$$\langle x - y, x - y\rangle < \epsilon$$

By definition of the hyperplane, we have

$$\langle y, \alpha\rangle = k$$

Thus,

$$\langle x, \alpha \rangle = \langle x - y + y, \alpha \rangle = \langle x - y, \alpha \rangle + \langle y, \alpha \rangle$$

Invoking the Cauchy-Schwartz inequality, it follows that

$$|\langle x, \alpha \rangle - k| < \epsilon |\alpha|$$

where $|\alpha|$ is the length of α. Since there are only finitely-many distinct roots α, if there were infinitely-many hyperplanes withing distance ϵ of x then for some root α_o there would be infinitely-many integers k so that

$$|\langle x, \alpha \rangle - k| < \epsilon |\alpha|$$

This is certainly impossible, contradicting the assumption that local finiteness fails.

The reflection in $\eta_{\alpha,k}$ is just $s_{\alpha,k}$, so the affine Weyl group W_a is the group corresponding to the locally finite collection H of affine hyperplanes. ♣

Corollary: This group W_a is an affine reflection group, so is a Coxeter group.

Proof: The local finiteness allows application of our earlier discussion of affine reflection groups generated by reflections in locally finite sets of hyperplanes, which we showed to be Coxeter groups, etc. ♣

As above, S is the set of generators s_α of W for $\alpha \in \Delta$. Recall that (W, S) is said to be *indecomposable* if S cannot be partitioned into two non-empty sets of mutually commuting generators. This assumption is equivalent to the indecomposability of the Coxeter matrix of (W, S), and to the connectedness of the Coxeter graph of (W, S).

Corollary: Still assume that Φ is a reduced finite crystallographic root system. If the Coxeter system (W, S) is *indecomposable*, then the affine reflection group W_a is generated by $n + 1$ reflections, including the n linear reflections $s_\alpha = s_{\alpha,0}$ for simple roots α. The chambers cut out are simplices.

Proof: The previous corollary's assertion, that W_a is a semi-direct product of a translation group and of W, shows that the point 0 is a *special* (or *good*) vertex for the affine reflection group W_a. That is, as in the previous section on affine reflection groups, the map from W_a to the group of linear parts of the maps is *surjective* when restricted to W.

Further, since W_a contains the coroot lattice, the chambers cut out by this affine reflection group have compact closure. Thus, by results on affine reflection groups, since W is indecomposable the chambers are simplices. (Note that our present W is the $\overline{W}$ of the previous section on affine reflection groups).

Since 0 is a *good* vertex (with stabilizer W), from our discussion of affine reflection groups in the previous section we know that there is a chamber C cut out by W_a with walls $\eta_{s,0}$ for $s \in S$, *and with just one more wall.* ♣

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13. Affine Coxeter Complexes

- Tits' cone model of Coxeter complexes
- Positive-definiteness: the spherical case
- A lemma from Perron-Frobenius
- Local finiteness of Tits' cones
- Definition of geometric realizations
- Geometric realization of affine Coxeter complexes
- The canonical metric
- The seven infinite families

The main goal here is to give a 'geometric realization' of Coxeter complexes, upon which we can put a metric structure, justifying to some degree both the appellations *'spherical'* and *'affine'*.

13.1 Tits' cone model of Coxeter complexes

Here we do preparatory work, giving Tits' construction which provides a link between abstract Coxeter complexes on one hand and 'concrete' reflection groups on the other hand. Specifically, we look further at the linear representation (1.3) of a Coxeter group on a finite-dimensional real vectorspace V, and follow Tits' construction of a poset of subsets of the dual V^* 'realizing' the Coxeter complex (3.4).

Let (W, S) be a Coxeter system (1.2) with associated Coxeter form $\langle , \rangle$ on the real vectorspace V with basis e_s for $s \in S$. We assume that S is finite, of cardinality n. We have the linear representation

$$W \stackrel{\sigma}{\to} \Gamma \subset G \subset GL(V)$$

defined on generators by

$$\sigma(s)(v) = v - 2\langle v, e_s\rangle e_s$$

where G is the isometry group of the (possibly degenerate) Coxeter form $\langle , \rangle$. In our earlier discussion (1.3), (1.4), (1.5) we saw that this map is an injection.

Let $\mathrm{GL}(V)$ have the 'usual' topology. This can be described in many equivalent ways. For example, the real-linear endomorphisms of V can be identified with the n^2-dimensional real vectorspace of real $n \times n$ matrices. The latter can be given the topology of R^{n^2}, and then $GL(V)$ given the subspace topology.

In any event, we give G the subspace topology it inherits from $GL(V)$. The isometry group G is, by definition, the subset of $\mathrm{GL}(V)$ consisting of elements g so that

$$\langle gv, gv'\rangle = \langle v, v'\rangle$$

for all v, v'. The function $\langle , \rangle$ is continuous, so these are 'closed conditions', so G is an intersection of closed subsets of $\mathrm{GL}(V)$, so is closed.

Let σ^* be the contragredient representation of W on the (real-linear) dual space V^* of V, defined 'as usual' by

$$(\sigma^*(w)\lambda)(v) = \lambda(\sigma(w^{-1})v)$$

We simplify notation by writing simply wv in place of $\sigma(w)v$, and also now write $w\lambda$ in place of $\sigma^*(\lambda)$.

A problem in using the Coxeter form to talk about the geometry on V is that it may be degenerate, and then not give an isomorphism of V with its real-linear dual. Therefore, for present purposes, *instead of the Coxeter bilinear form on $V \times V$, we use the canonical bilinear pairing*

$$\langle , \rangle : V \times V^* \to \mathbb{R}$$

That is, for $v \in V$ and $x \in V^*$, we now use notation

$$\langle v, x \rangle = x(v)$$

For $s \in S$ define *walls, upper half-spaces (half-apartments), and lower half-spaces* (respectively) in V^* by

$$Z_s = \{x \in V^* : \langle e_s, \lambda \rangle = 0\}$$

$$A_s = \{x \in V^* : \langle e_s, x \rangle > 0\}$$

$$B_s = \{x \in V^* : \langle e_s, x \rangle < 0\} = sA_s$$

and the *fundamental chamber*

$$C = \bigcap_{s \in S} A_s$$

The sets A_s and B_s are open, and Z_s is closed. Note that, since S is finite, C is a finite intersection of opens, so is open. Also, s interchanges A_s and B_s, and fixes Z_s pointwise; indeed, Z_s is visibly the fixed-point set of s.

For a subset I of S, let

$$F_I = \left(\bigcap_{s \in I} Z_s \right) \cap \left(\bigcap_{s \notin I} A_s \right)$$

Then $F_\emptyset = C$ and $F_S = \{0\}$. Let W_I be special subgroup (1.9) of W generated by I. We observe that $F_I \neq \emptyset$, as follows. Let $\{x_s\}$ be a basis for V^* dual to the basis $\{e_s\}$ for V, that is,

$$\langle e_s, x_t \rangle = 0 \text{ for } \quad s \neq t$$

and $\langle e_s, x_s \rangle = 1$. Then

$$\sum_{s \notin I} x_s$$

is visibly in F_I.

It is easy to see that the group W_I fixes F_I pointwise: each $s \in I$ fixes Z_s pointwise, so certainly $F_I \subset Z_s$ is fixed pointwise by the subgroup W_I of W generated by I.

On the other hand, if $s \in S$ does fix $\lambda \in F_I$, then

$$\langle e_s, \lambda \rangle = \langle se_s, s\lambda \rangle =$$

$$= -\langle e_s, s\lambda \rangle = -\langle e_s, \lambda \rangle$$

so $\lambda \in Z_s$. Thus, if s fixes *every* $\lambda \in F_I$, then since the e_t for $t \in I$ are linearly independent, it must be that $s \in I$. It is not yet clear, however, that W_I is *exactly* the stabilizer of *every* point in F_I.

Define the **Tits' cone**

$$U = \bigcup_{w \in W} w\overline{C}$$

where

$$\overline{C} = \bigsqcup_I F_I$$

is the topological closure of C.

Theorem: The Tits' cone U is a convex cone in V^*, and every closed line segment in U meets only finitely-many sets of the form wF_I. If $wF_I \cap F_J \neq \emptyset$, then $I = J$ and $w \in I$ (so actually $wF_J = F_I$ and $J = I$ and $w \in W_I$). The set $\overline{C}$ is a fundamental domain for W acting on U. That is, given a point $u \in U$, the W-orbit Wu of u meets $\overline{C}$ in exactly one point.

Proof: First, the fact proven earlier (1.4), (1.5) that $\ell(ws) > \ell(w)$ if and only if $we_s > 0$ (etc.) can be immediately paraphrased as follows: for $s \in S$ and $w \in W$,

$$\ell(sw) > \ell(w) \iff wC \subset A_s$$

$$\ell(sw) < \ell(w) \iff wC \subset B_s$$

Note that we consider sw rather than ws.

We first prove the assertion concerning $wF_I \cap F_J$, by induction on $\ell(w)$. If $\ell(w) = 0$ then we are done. If $\ell(w) > 0$, then there is $s \in S$ so that $\ell(sw) < \ell(w)$. As just noted, this implies that $wC \subset sA_s = B_s$. By continuity, $w\overline{C} \subset \overline{B}_s$, where $\overline{B}_s$ is the topological closure of B_s. Since $F_I \subset \overline{C} \subset \overline{A}_s$, we have $\overline{C} \cap w\overline{C} \subset Z_s$. Therefore, s fixes each point in the assumedly non-empty set $wF_I \cap F_J$.

Since s fixes *some* point of F_J, from the short remarks preceding the theorem we have $s \in J$. Also,

$$swF_I \cap F_J = s(wF_I \cap F_J) \neq \emptyset$$

Thus, induction applied to sw implies that $I = J$ and $sw \in W_I$. Since $s \in J = I$, it must be that $w \in W_I$.

Thus, we find that the sets wF_I are disjoint for distinct cosets wW_I and distinct subsets $I \subset S$. This gives the second assertion of the theorem.

From the definition of the 'cone' U, each W-orbit meets $\overline{C}$ in at least one point. Suppose that $\lambda, \mu \in \overline{C}$ are both in the same W-orbit: take $w \in W$ so that $w\lambda = \mu$. Take $I, J \subset S$ so that $\lambda \in F_I$ and $\mu \in F_J$. Then $wF_I \cap F_J \neq \emptyset$

implies that $I = J$ and $w \in W_I$, so $\lambda = \mu$. This proves that $\overline{C}$ is a fundamental domain for the action of W on U.

Next show that U is a convex cone: from the definition, it is immediate that U is closed under taking positive real multiples. Thus, it suffices to show that, for $\lambda, \mu \in U$, the closed line segment $[\lambda, \mu]$ connecting them lies inside U. In fact, we will prove that it is covered by finitely-many of the (disjoint) sets wF_I.

The assertion is clear if λ, μ are in $\overline{C}$, which is convex and covered by the F_I's, of which there are finitely-many since S is finite.

Without loss of generality, take $\lambda \in \overline{C}$ and $\mu \in w\overline{C}$. We do induction on $\ell(w)$, considering only $\ell(w) > 0$. Now $[\lambda, \mu] \cap \overline{C} = [\lambda, \nu]$ for some $\nu \in \overline{C}$, so is covered by finitely-many of the disjoint sets F_I. Since $\mu \notin \overline{C}$, there is $I \subset S$ so that $\mu \in B_s$ for $s \in I$ and $\mu \in \overline{A_s}$ for $s \notin I$. If ν were in A_s for all s in I, then other points on $[\nu, \mu]$ close to ν would also be in A_s for $s \in I$ and in $\overline{A}_s$ for $s \notin I$, since both $\mu, \nu \in \overline{A}_s$ for $s \notin I$. But then such points near ν would also lie in $\overline{C}$, contradicting the definition of ν (and the convexity of $\overline{C}$).

Therefore, for some $s \in I$, $\nu \in Z_s$. Since $\mu \in B_s$, $w\overline{C} \subset \overline{B}_s$. Then $wC \subset B_s$. The first remarks of this proof yield $\ell(sw) < \ell(w)$. By induction on length, $s\nu = \nu \in \overline{C}$ and $s\mu \in sw\overline{C}$, and also $[\nu, s\mu]$ has a finite cover by sets $w'F_J$. From this the assertion of the theorem follows. ♣

Corollary: The image Γ of W in the isometry group G is a *discrete* (closed) subgroup of G.

Proof: Fix $\lambda \in C$. The map $\Lambda : w \to w\lambda$ is continuous, so since C is open $N = \Lambda^{-1}(C)$ is open. Certainly N contains 1. The theorem shows that $\sigma^*(W) \cap N = \{1\}$.

This suffices to prove that the image of W in $GL(V^*)$ is discrete, as follows. Let N' be a neighborhood of 1 so that $xy^{-1} \in N$ for all $x, y \in N'$; this is possible simply by the continuity of multiplication and inverse. If a sequence γ_i of images of elements of W in $GL(V^*)$ had a limit point h, then for sufficiently high index i_o we would have $\gamma_i^{-1}h \in N'$ for all $i > i_o$. Then $\gamma_i\gamma_j^{-1} \in N$ for all $i, j > i_o$. Since N meets the image of W just at 1, this shows that $\gamma_i\gamma_j = 1$ for all $i, j > i_o$. Thus, discreteness is proven.

From this, we see that the image of W in $GL(V)$ is discrete, since the 'adjoint' map by $g \to g^*$ defined by

$$\langle v, g^*\lambda \rangle = \langle gv, \lambda \rangle$$

is readily seen to be a homeomorphism of $GL(V)$ to $GL(V^*)$. ♣

Now let X be the poset of sets wF_I, where we use the ordering that

$$wF_I \leq w'F_J$$

if wF_I is in the topological closure of $w'F_J$ in the usual topology on V.

Corollary: The poset of sets wF_I (with $I \neq S$) with ordering just described is isomorphic (as 'abstract' simplicial complex) to the Coxeter complex of (W, S), via

$$\phi : wF_I \to wW_I$$

where W_I is the subgroup of W generated by a subset I of S. Further, this isomorphism respects the action of W, in the sense that

$$\phi(w\sigma) = w\phi(\sigma)$$

for all $w \in W$ and for all simplices σ.

Proof: Again, the Coxeter complex was described as a poset and as a simplicial complex in (3.4).

First, the requirement that $I \neq S$ removes 0 from U. This is certainly necessary for there to be such a poset isomorphism, since otherwise $\{0\} = F_S$ would be the unique minimal simplex in the complex, which is absurd.

In the theorem we showed that if two sets wF_I and $w'F_J$ have non-empty intersection then $w = w'$ and $I = J$. Thus, certainly ϕ is well-defined. Then it certainly is a bijection of sets, and visibly respects the action of W.

If the closure of wF_I contains $w'F_J$, then the closure of F_I contains $w^{-1}w'F_J$. Then by the theorem $w^{-1}w'F_J$ is of the form F_K for some $K \subset S$, so necessarily $w^{-1}w' \in W_J$, the stabilizer of F_J. and $w^{-1}w'F_J = F_J$. Then $I \subset J$. This shows that ϕ preserves inequalities.

On the other hand, if $wW_I \geq w'W_J$ then by definition $wW_I \subset w'W_J$. Then $W_I \subset w^{-1}w'W_J$. Since W_I is a subgroup of $W_J \subset W$, it must be that $w^{-1}w' \in W_J$, and $I \subset J$. Then the reverse of the argument of the previous paragraph shows that wF_I contains $w'F_J$. ♣

13.2 Positive definiteness: the spherical case

Throughout this section, the standing assumption on the Coxeter system (W, S) is that the Coxeter form is *positive definite*. All we want to do is prove that this implies that the Coxeter group is a finite linear reflection group (12.3), although one can continue easily in this vein, for example proving that the Coxeter complex is a triangulation of a sphere.

Corollary: If the Coxeter form is positive-definite then the group W is a *finite* group, and consists of linear reflections.

Proof: If the Coxeter form is positive-definite, then the isometry group G of it is *compact*, being the orthogonal group (that is, isometry group) attached to a *positive-definite* quadratic form over the real numbers. (This is a standard sort of fact, and is a worthwhile elementary exercise to consider). From above, the image under the linear representation σ of the Coxeter group W in $GL(V)$ is a *discrete* (closed) subgroup. (We saw much earlier that the map $W \to GL(V)$ is injective). A discrete (closed) subset of a compact set is finite.

Since it is positive-definite, the Coxeter form gives an *inner product* on the space V. By construction, the images $\sigma(s)$ for $s \in S$ are *orthogonal reflections* with respect to the inner product arising from the positive-definite Coxeter form. Let η_s be the linear hyperplane fixed by $\sigma(s)$. Since, as we have just seen, the whole group is finite, there must be only finitely-many hyperplanes $w\eta_s$ for $w \in W, s \in S$. Since, after all, W is generated by S, it must be that $\sigma(W)$ is the *finite linear reflection group* generated by the $\sigma(s)$. ♣

Remarks: Further, since V has basis consisting of vectors e_s which are -1 eigenvalues for $\sigma(s)$, it is clear that the action of W on V is *essential* (12.3).

13.3 A lemma from Perron-Frobenius

Here is a prerequisite to the affine case, which is a bit of peculiar elementary linear algebra. This is a small part of what is apparently called 'the Perron-Frobenius theory of non-negative matrices and M-matrices'.

A symmetric $n \times n$ matrix is sometimes called *indecomposable* (compare (1.2)) if there is *no* partition $\{1, \ldots, n\} = I \sqcup J$ of the index set into non-empty subsets so that the $(i,j)^{\text{th}}$ entry M_{ij} is 0 for $i \in I$ and $j \in J$.

In the sequel, we will concern ourselves with Coxeter systems (W, S) whose Coxeter matrix meets the hypotheses of the following elementary lemma, whose conclusion will allow us to see (a little later) that the associated Coxeter complex 'is' an *affine space*.

Recall that a symmetric bilinear form $\langle , \rangle$ is *positive semi-definite* if $\langle v, v\rangle \geq 0$ for all $v \in V$, and *positive-definite* if $\langle v, v\rangle = 0$ implies that $v = 0$.

Lemma: Let M be an indecomposable real symmetric $n \times n$ matrix which is positive semi-definite. Assume further that $M_{ij} \leq 0$ for $i \neq j$. Then

$$\{v \in \mathbb{R}^n : Mv = 0\} = \{v \in \mathbb{R}^n : v^\top Mv = 0\}$$

where $v^\top$ is the transpose of v and we view v as a column vector. Further, the dimension of the kernel $\{v \in \mathbb{R}^n : v^\top Mv = 0\}$ of M is 1. Finally, the smallest eigenvalue of M has multiplicity one, and has an eigenvector with all positive coordinates.

Proof: The inclusion

$$\{v \in \mathbb{R}^n : Mv = 0\} \subset \{v \in \mathbb{R}^n : v^\top Mv = 0\}$$

is clear.

Since M is symmetric an positive semi-definite, by the spectral theorem there is an orthogonal matrix Q so that $QMQ^\top = D$ is diagonal with non-negative diagonal entries $D_1, \ldots, D_n$. Then

$$v^\top Mv = v^\top Q^\top DQv = (Qv)^\top D(Qv) = \sum_i D_i w_i^2$$

where w_i is the i^{th} coordinate of $w = Qv$. By the non-negativity of the D_i, if $v^\top Mv = 0$, then it must be that for each index i we have $D_i w_i^2 = 0$, so either $D_i = 0$ or $w_i = 0$. Then immediately $Dw = 0$. Thus,

$$0 = Q^\top \cdot 0 = Q^\top Dw = Q^\top D(Qv) = Mv$$

Thus, we have equality of the two sets.

Suppose that the kernel of M has positive dimension. Take $0 \neq v$ in the kernel. Let u be the vector whose entries are the absolute values of those of v. Since $M_{ij} \leq 0$ for $i \neq j$, we obtain the second inequality in the following:

$$0 \leq u^\top Mu \leq v^\top Mv = 0$$

Thus, u also lies in the kernel.

Now we show that all coordinates of u are non-zero. Let J be the non-empty set of indices so that $u_j \neq 0$ for $j \in J$, and let I be its complement. Since

$$\sum_j M_{ij} u_j = \sum_{j \in J} M_{ij} u_j = 0$$

for all indices $i \in I$, and since $M_{ij} \leq 0$, for $j \in J$ and $i \in I$ we have $M_{ij} u_j \leq 0$. Since the sum is 0, each non-positive summand $M_{ij} u_j$ (with $j \in J$ and $i \in I$) is actually 0. If I were non-empty, this would contradict the indecomposability of M. Thus, $I = \emptyset$, so u has all *strictly* positive coordinates.

Since u was made by taking absolute values of an arbitrary vector v in the kernel, this argument shows that *every non-zero vector in the kernel of M has all non-zero entries.* This precludes the possibility that the dimension of the kernel be larger than 1: if the dimension were two or larger, a suitable non-zero linear combination of two linearly independent vectors can be arranged so as to have some entry zero.

Let d be the smallest (necessarily non-negative) eigenvalue of M. Let I be the identity matrix of the same size as M. Then $M - dI$ still satisfies the hypotheses of the lemma, and now has an eigenvalue zero. Thus, as we just proved, its zero eigenspace has dimension one, so the d-eigenspace of M has dimension one. ♣

Corollary: Let M be an indecomposable real symmetric $n \times n$ matrix which is positive semi-definite. Assume further that $M_{ij} \leq 0$ for $i \neq j$. Let N be the $(n-1) \times (n-1)$ matrix obtained by deleting the i_o^{th} row and column from M. Then N is positive-definite.

Proof: Without loss of generality, take $i_o = 1$. If N were not positive definite, then there would be a non-zero vector $w = (w_2, \ldots, w_n)$ in $\mathbb{R}^{n-1}$ so that $w^\top N w \leq 0$. Let

$$v = (0, |w_2|, \ldots, |w_n|)$$

Then, letting M_{jk} be the $(j,k)^{\text{th}}$ entry of M,

$$0 \leq v^\top M v \leq = \sum_{i,j} M_{ij} |w_i||w_j| \leq$$

$$\leq \sum_{i,j} M_{ij} w_i w_j$$

since $M_{ij} \leq 0$ for $i \neq j$. Thus, we have

$$0 \leq \sum_{i,j} M_{ij} w_i w_j \leq w^\top N w \leq 0$$

Thus, equality holds throughout, and by the lemma $Mv = 0$. But the lemma also asserts that $Mv = 0$ implies that all entries of v are non-zero (if $v \neq 0$), contradiction. Thus, N is positive definite. ♣

13.4 Local finiteness of Tits' cones

For this section, to prove the desired local finiteness properties of Tits' cone model (13.1), we will have to assume that the Coxeter groups W_I for I a subset of S with $I \neq S$ are *finite*.

We will see that this hypothesis is met in the case that (W, S) is **affine**, which by definition means that it has indecomposable Coxeter matrix which is positive semi-definite but not positive definite. (Again, this terminology will be justified a little later).

Recall (1.4) that the collection Φ of all *roots* of (W, S) is

$$\Phi = \{we_s : w \in W, s \in S\}$$

For present purposes, we suppose that all the roots $w\alpha = e_s$ are 'unit vectors' in the sense that $\langle \alpha, \alpha \rangle = 1$, where $\langle , \rangle$ is the Coxeter form.

The set Φ^+ of *positive roots* is the collection of those roots which are non-negative real linear combinations of the roots e_s. The set Δ of the roots e_s (for $s \in S$) is the set of *simple* roots. We saw earlier (1.4), (1.5) that always

$$\Phi = \Phi^+ \sqcup -\Phi^+$$

Also, for $w \in W$, the length $\ell(w)$ of w is equal to the number of positive roots α so that $w\alpha$ is negative (that is, is the negative of a positive root).

From the earlier discussion of roots, we know that the fundamental chamber C as defined earlier is

$$C = \{x \in V^* : \langle \alpha, x \rangle > 0 \quad \forall \alpha \in \Phi^+\}$$

And a root β is positive if and only if $\langle \beta, x\rangle > 0$ for all $x \in C$ (or, equivalently, for *one* $x \in C$).

For $x \in V^*$, let $\nu(x) \le \infty$ be the number of $\alpha \in \Phi^+$ so that

$$\langle \alpha, x\rangle \le 0$$

For example, the fundamental chamber C is the subset of V^* where $\nu = 0$.

Lemma: If W_I is *finite* for subsets I of S strictly smaller than S, then the Tits' cone U associated to (W, S) is

$$U = \{0\} \cup \{x \in V^* : \nu(x) < \infty\}$$

Proof: Let X be the set of x with $\nu(x) < \infty$.

First, from the fact that $\ell(w)$ is the number of positive roots taken to negative, it follows that for $x \in C$ and $w \in W$ we have

$$\nu(wx) = \ell(w) < \infty$$

Thus, all the images wC of the fundamental chamber C lie inside X.

Next, we check that each F_I with $I \ne S$ lies inside X. Thus, there are only finitely-many positive roots which are linear combinations of just $\{e_s : s \in I\}$. Thus, since $\langle e_s, x\rangle > 0$ for $x \in F_I$ and $s \notin I$, generally $\langle \alpha, x\rangle > 0$ for positive roots α which are not linear combinations of just $\{e_s : s \in I\}$. This shows that for $x \in F_I$ with $I \ne S$ we have $\nu(x) < \infty$, so $F_I \subset X$, as claimed.

Last, we show that $wF_I \subset X$ for $w \in W$ and $I \ne S$. This argument just is a slight extension of that for the case $F_I = C$. Take $x \in F_I$. For $\alpha \in \Phi^+$, consider the condition

$$0 \ge \langle \alpha, wx\rangle = \langle w^{-1}\alpha, x\rangle$$

This condition requires that either $w^{-1}\alpha$ be a negative root, or among the *finite* number of positive roots β so that $\langle \beta, x\rangle = 0$ on F_I. Since $\ell(w)$ is finite and is the number of positive roots sent to negative, there are altogether only finitely-many positive roots α so that

$$0 \ge \langle \alpha, wx\rangle$$

That is, $wF_I \subset X$ for $I \ne S$. ♣

Remarks: The point of the following lemma is that the hypotheses of the preceding lemma are indeed satisfied by affine systems (W, S). After all, these affine Coxeter systems are the main object of interest here.

Lemma: For affine (W, S) the subgroups W_I *are finite* for a proper subset I of S.

Proof: Indeed, by the Perron-Frobenius lemma (13.3) the Coxeter matrix for a proper subset I of S is *positive definite*. From our discussion (13.2) of the case where the Coxeter matrix is positive definite (that is, 'the spherical case'), we know that the Coxeter group W_I is indeed a *finite* group, since it is a *discrete* subgroup of a *compact* isometry group. ♣

Corollary: For (W, S) so that W_I is finite for proper subsets I of S, the set $U - \{0\}$ (that is, the Tits' cone with 0 removed) is an open subset of the ambient space V^*. Further, the W-stable set H of linear hyperplanes

$$\eta_{s,w} = \{x \in V^* : \langle e_s, wx \rangle = 0\}$$

is *locally finite in* $U - \{0\}$, in the sense that for a compact set K inside $U - \{0\}$, there are only finitely-many $\eta \in H$ so that $K \cap \eta \neq \emptyset$.

Proof: Let Ψ be the set of subsets Φ' of Φ^+ so that the difference $\Phi^+ - \Phi'$ is *finite*. For $\Phi' \in \Psi$, let

$$U(\Phi') = \{x \in V^* : \langle \alpha, x \rangle > 0, \ \forall \alpha \in \Phi'\}$$

Then each $U(\Phi')$ is visibly open, and

$$U - \{0\} = \bigcup_{\Phi' \in \Psi} U(\Phi')$$

is a union of opens, so is open.

To prove the asserted local finiteness, for elementary reasons we need only consider compact sets K which are the closed convex hulls of $n + 1$ points of U, where $n = \dim V^*$. If a linear hyperplane

$$\eta = \{x \in V^* : \langle v, x \rangle = 0\}$$

meets such a set K, then there must be a pair y, z of vertices of K so that $\langle v, y \rangle$ and $\langle v, z \rangle$ are not both > 0 and not both < 0; otherwise, taking convex combinations, the linear function $x \to \langle v, x \rangle$ would be > 0 or < 0 (respectively) on the whole set K. Thus, if η meets K, then η must meet a line segment ℓ connecting two vertices of K. Of course, there are only finitely-many such line segments for a give set K of this form.

Thus, it suffices to show that a closed line segment ℓ inside $U - \{0\}$ meets only finitely-many of the hyperplanes $\eta_{s,w}$. Now $\eta_{s,w} \cap U$ is the union of sets $w'F_I$. We showed that in general a line segment ℓ inside the Tits' cone meets only finitely-many sets $w'F_I$. This gives the local finiteness. ♣

Remarks: Without the assumption that W_I is finite for I a proper subset of S, the lemma and corollary are false, although some parts of the arguments still go through.

It is still true in general, by the same argument as just above, that the set X where ν is finite is open and contains all images wC, but it is *not* true that F_I lies inside X if it should happen that W_I is infinite. This is because if W_I is infinite then there must be infinitely-many positive roots which are linear combinations of e_s for $s \in I$, so then F_I lies on infinitely-many hyperplanes $\langle we_s, x \rangle = 0$, so is not in X.

It is still true in general that the set X, which *is* obviously an open set even if U may not be, is a convex W-stable cone, by the same argument as in the section on Tits' cones in general. But in general we do *not* obtain a model for the Coxeter complex, since we will have lost those faces F_I with infinite isotropy groups W_I.

It is still true, by the same argument as just above, that the set of hyperplanes $\eta_{s,w}$ is locally finite in the set X. But this is a far weaker assertion (in general) than the assertion of local finiteness in U, which may contain limit points of X, for example.

13.5 Definition of geometric realizations

We need a notion of *geometric realization* of a simplicial complex X. This section is essentially elementary and standard, establishing some necessary conventions.

Recall that a *(combinatorial) simplicial complex* is a set X of subsets (called *simplices*) of a *vertex set* V, so that if $x \in X$ and $y \subset x$ then $y \in X$.

Define the geometric realization $|X|$ to be the collection of non-negative real-valued functions f on the vertex set V of X so that

$$\sum_{v \in V} f(v) = 1$$

and so that there is $x \in X$ so that $f(v) \neq 0$ implies $v \in x$.

For example, if X is a *simplex* (that is, is the set of all subsets of V), then we imagine $|x|$ to be the collection of 'affine combinations' of the vertices.

Recall that a map $\phi : X \to Y$ is a map of the vertex sets so that for every simplex $x \in X$ the image $\phi(x)$ is a simplex in Y. We will only consider maps $\phi : X \to Y$ of simplicial complexes so that for all simplices $x \in X$ the restriction ϕ_x is *injective*. In particular, we are requiring that ϕ *preserve dimension of simplices*. This is part of the definition of the *chamber complex* maps we considered earlier.

For such ϕ, the restriction $\phi|_x$ to a simplex $x \in X$ is *invertible*, since it is injective. Then we have a *natural geometric realization* $|\phi|$ of the map ϕ, given by

$$|\phi| : |X| \to |Y|$$

on the geometric realizations, defined as follows. For $f \in |X|$, let $x \in X$ be such that f is 0 off x. Then for v' in the vertex set of Y, put

$$|\phi|(f)(v') = f(\phi^{-1}v') \quad \text{for} \quad v' \in \phi(x)$$

$$|\phi|(f)(v') = 0 \qquad \text{for} \quad v' \notin \phi(x)$$

The topology on $|X|$ can be given by a metric, as follows. For $f, g \in |X|$, define the distance $d(f, g)$ between them by

$$d(f,g) = \sup_{v \in V} |f(v) - g(v)|$$

where V is the vertex set of X.

It is immediate that the geometric realization $|\phi|$ of a simplicial complex map $\phi : X \to Y$ (whose restrictions to all simplices are injective) has the property that

$$d_X(f,g) \geq d_Y(|\phi|(f), |\phi|(g))$$

From this it is clear that *the geometric realization $|\phi|$ is a continuous map of topological spaces.*

In particular, for a simplex $x \in X$, we have a continuous inclusion $|x| \to |X|$. And it is clear that the geometric realization of a ('combinatorial') simplex $x = \{v_o, \ldots, v_m\}$ is a ('geometric') simplex

$$\{(t_o, \ldots, t_m) : 0 \leq t_i \leq 1 \text{ and } \sum_i t_i = 1\}$$

The map is the obvious one:

$$f \to (f(t_o), \ldots, f(t_m))$$

Very often one is presented with a vertex set imbedded in a real vectorspace Z, and one wants to have the geometric realization $|X|$ be 'imbedded' in Z. Let $i : V \to Z$ be a set map of the vertex set to Z. For a simplex $x = \{v_0, \ldots, v_m\}$ in X, let $i(x)^o$ denote the set of convex combinations

$$t_o i(v_o) + \ldots + t_m i(v_m)$$

where $0 < t_j < 1$ for all indices j. This is the *open convex hull* of the point set $i(x)$. We can define the 'obvious' map

$$|i| : |X| \to Z$$

as follows: for $f \in |X|$ which is zero off a simplex $x = \{v_0, \ldots, v_m\}$, let

$$|i|(f) = f(v_o)i(v_o) + \ldots + f(v_m)i(v_o)$$

It is easy to check that such a map $|i|$ is *continuous.*

And clearly *$|i|$ is injective if and only if for any two simplices x, y of X if $i(x)^o \cap i(y)^o \neq \emptyset$ then $x = y$.* In particular, this condition implies that, for a simplex $x = \{v_o, \ldots, v_m\}$ in X, the images $i(v_o), \ldots, i(v_m)$ are *affinely independent.* In particular, if there is such an injection $|X| \to Z$, it must be that the dimension of Z is greater than or equal the dimension of the simplicial complex X.

Recall that a set Ω of subsets of Z is *locally finite* if any compact subset of Z meets only finitely many sets in Ω.

Lemma: If $|i| : |X| \to Z$ is injective, and if the set Ω of images $|i|(|x|)$ of geometric realizations of simplices x in X is *locally finite* in Z, then $|i|$ is a *homeomorphism* of $|X|$ to its image.

Proof: We must prove that the inverse of $|i|$ (which exists because $|i|$ is assumed injective) is continuous on $|i|(|X|)$. To this end, the local finiteness allows us to assume without loss of generality that there are only finitely-many simplices in X altogether. Then $|X|$ is compact, since it is a finite union of geometric realizations $|x|$ of its simplices, and these are compact sets. Thus, we have a continuous injection $|i|$ of the compact topological space $|X|$ to the Hausdorff topological space $|i|(|X|) \subset Z$.

A standard and elementary point-set topology argument shows that $|i|$ is a homeomorphism, as follows: let U be open in $|X|$. Then $C = |X| - U$ is a closed subset of a compact space, so is compact. Thus, the continuous image $|i|(C)$ is compact, so is closed since $|i|(|X|)$ is Hausdorff. ♣

13.6 Criterion for affineness

Here we finally prove that if the Coxeter form is *affine* then the geometric realization *really is a Euclidean space*. Further, the Coxeter group acts as an affine reflection group, and the chambers cut out are n-simplices.

Thus, our *definition* (via indecomposability and positive semi-definiteness, etc.) is really the *criterion*, but what have delayed proof until now. It is only now that justification is provided for the term *affine*, even though it has been used for a while.

Remarks: Here 'simplex' is used in the 'physical' sense as in the discussion of reflection groups, rather than in the 'combinatorial' sense as in the discussion of simplicial complexes).

By definition (13.4), a Coxeter system (W, S) is **affine** if it is indecomposable and if the associated Coxeter matrix is positive semi-definite but not positive definite.

Remarks: Any Coxeter system (W, S) can be written as a 'product' of indecomposables in the obvious manner, so there is no loss of generality in treating indecomposable ones. And, the assumption of indecomposability is *necessary* to obtain the cleanest results.

Let (W, S) be *affine*. As usual, let e_s for $s \in S$ be the basis for the real vectorspace V on which W acts by the linear representation σ. We identify $w \in W$ with its image by σ. We have the contragredient representation σ^* on the dual space V^*. In either case we identify W with its image in the group of automorphisms. This is reasonable since we have already shown that W *injects* to its image.

Let

$$V^\perp = \{v \in V : \langle v, v' \rangle = 0 \ \ \forall v' \in V\}$$

Then on the quotient $V/V^\perp$ the symmetric bilinear form induced from $\langle , \rangle$ becomes *positive definite*. Since $V^\perp$ is the intersection of all hyperplanes

$$H_s = \{v \in V : \langle v, e_s \rangle = 0\}$$

it is W-stable.

By the Perron-Frobenius lemma (13.3), under our present hypotheses, the subspace $V^\perp$ is one-dimensional, and is spanned by a vector $v_o = \sum_s c_s e_s$ with all coefficients c_s *positive*.

Thus, under the contragredient action σ^* of W on the dual space V^*, the group W stabilizes

$$Z = \{\lambda \in V^* : (v, \lambda) = 0 \quad \forall v \in V^\perp\}$$

where $(,)$ is the canonical pairing $V \times V^* \to \mathbb{R}$. This gives a standard identification

$$Z \approx (V/V^\perp)^*$$

by $z \to \lambda_z$ with

$$\lambda_z(v + V^\perp) = (v, z)$$

Since the form (still written as $\langle , \rangle$) induced on $V/V^\perp$ from $\langle , \rangle$ is non-degenerate, it gives a natural vectorspace isomorphism of $V/V^\perp$ with its dual Z, by $v + V^\perp \to \lambda_v$ with

$$\lambda_v(v') = \langle v', v\rangle$$

Thus, via this natural isomorphism, the positive definite form induced by $\langle , \rangle$ on $V/V^\perp$ induces a positive definite form on Z in a natural way.

Let

$$E = \{\lambda \in V^* : (v_o, \lambda) = 1\}$$

This *affine* subspace of V^* is a translate of Z by any λ_o so that $(v_o, \lambda_o) = 1$. The group W stabilizes E under the action via σ^*, since W fixes v_o. The *linear* automorphisms $\sigma^*(w)$ of V^* give rise to *'affine'* automorphisms of E, simply by restriction. In particular, W fixes λ_o and preserves the inner product.

We use the notation from our discussion of Tits' cones. Since v_o has all positive coefficients when expressed in terms of the e_s, v_o and e_s are not parallel (noting that necessarily $\mathrm{card}(S) > 1$). Thus, the set

$$\eta_s = Z_s \cap E = \{\lambda \in V^* : (e_s, \lambda) = 0 \ \text{ and } \ (v_o, \lambda) = 1\}$$

is an affine hyperplane in E, as opposed to being empty or being all of E.

Depending on the choice of λ_o, the positive definite symmetric bilinear form $\langle , \rangle$ on Z can be 'transported' to a form $\langle , \rangle_E$ on E by

$$\langle \lambda, \lambda'\rangle_E = \langle \lambda - \lambda_o, \lambda' - \lambda_o\rangle$$

Then a direct computation shows that $s \in S$ gives the *orthogonal reflection through the affine hyperplane* η_s, as affine automorphism of E. Note that the group W acts by *isometries* on E, where the metric is that obtained from $\langle , \rangle_E$:

$$d(x, y) = \langle x - y, x - y\rangle_E^{1/2}$$

The images $w\eta_s$ are evidently affine hyperplanes in E, as well. The set

$$H = \{w\eta_s : w \in W \ \text{ and } \ s \in S\}$$

is a W-stable set of affine hyperplanes in E.

In the Tits' cone notation, we are taking

$$\eta_s = Z_s \cap E$$

Let U be the Tits' cone

$$U = \bigcup_{w,I} wF_I = \bigsqcup_{w,I} wF_I$$

Remarks: The assertion of the following lemma seems obvious, but is false without *some* hypotheses. And the argument given in the proof below is not the most general, since we use extra information available here. In particular, we use the fact that all the *proper* 'special' subgroups of an affine Coxeter group are *finite*. That this is so uses the reflection group discussion, as well as the Perron-Frobenius lemma. (We used these same facts in a crucial way in obtaining finer results on the Tits' cone in this case). Still, this greatly simplifies the proof of the lemma.

Lemma: Assuming that (W, S) is affine, the set

$$E \cap U = \bigcup_{w,I} wF_I \cap E = \bigsqcup_{w,I} wF_I \cap E$$

is actually all of E.

Proof: Let σ be the n-simplex which is the closure of $F_\emptyset \cap E$. We may identify S with the collection of reflections through the affine hyperplanes $\eta_s = E \cap Z_s$ in E and identify W with the group of *isometries* of the affine space E generated by S. It is because of the nature of M that W acts by affine *isometries*.

Thus, we are claiming that

$$E = \bigcup_{w \in W} w\sigma$$

A more specific version of this assertion is easier to verify. Fix x_o in the interior of σ. Let H be the set of all hyperplanes $w\eta_s$ for $w \in W$ and $s \in S$. Take $x \in E$ but not lying in any of the hyperplanes $\eta \in H$, and not lying in any of the hyperplanes which contain both x_o and the intersection of two of the $\eta \in H$.

Then either $x \in \sigma$ or else the line segment $[x_o, x]$ from x_o to x meets the boundary $\partial\sigma$ of σ at a unique point x_1. Let t_1 be the reflection through the facet of σ containing x_1 and put $\sigma_1 = w_1\sigma$. Then either $x \in \sigma_1$ or the line segment $[x_1, x]$ meets the $\partial\sigma_1$ at a unique point x_1. Continuing inductively, we define $\sigma_m = w_m\sigma_{m-1}$.

We claim that for sufficiently large m the n-simplex σ_m contains x. This would prove the lemma, for the following reasons. The collection of hyperplanes x on which x cannot lie is *countable*, so the union of these hyperplanes is *nowhere dense* in E. (This elementary point was made in detail in our discussion of reflection groups). Thus, we are considering x in a dense subset of E. The Tits' cone U is convex (13.1), so $E \cap U$ is convex. Therefore, if we prove this claim, we will know that $E \cap U$ contains the convex closure of an everywhere dense subset of E, hence is all of E.

To prove the claim, it suffices to show that there is a number h and a number $\alpha > 0$ so that for indices j, k with $|j-k| > h$ the length of the line segment $[x_j, x_k]$ must be at least α. Moving everything by an element of W which takes σ_j back so σ, we need only consider the case $j = 0$. Here we use the fact that W acts by *isometries* of E.

Fix $s_o \in S$. We will first show that there is $h_{s_o} < \infty$ so that if $[x_o, x_i]$ meets no $w\eta_{s_o}$ (for $w \in W$) then $i \leq h_{s_o}$.

To see this, first observe that if $x_o, x_1, \ldots, x_i$ lie on no image $w\eta_{s_o}$ of η_{s_o} then all the reflections $t_o, t_1, \ldots, t_i$ are actually in the subgroup W_{s_o} of W generated by $S_{s_o} = S - \{s_o\}$. That is, all the intersection points $x_o, x_1, \ldots, x_i$ lie on hyperplanes of the form $w'\eta'$ where $w' \in W_{s_o}$ and $\eta' = \eta_s$ with $s \in S_{s_o}$.

The Coxeter matrix of W_{s_o} is positive definite, by the corollary to the Perron-Frobenius lemma above. Therefore, from our discussion of the 'spherical' case (13.2), the group W_{s_o} is *finite*. Therefore, the number h_{s_o} of hyperplanes $w'\eta'$ is also finite, bounded by the product of $\mathrm{card}(W_{s_o})$ and $\mathrm{card}(S_{s_o})$.

Take h to be the maximum of the numbers h_{s_o} as s_o ranges over S. We have shown that if $|j-k| > h$ then $[x_j, x_k]$ touches an image by W of every one of the hyperplanes η_s for $s \in S$.

Next we show that there is $\alpha > 0$ so that a line segment $[x_j, x_k]$ which touches an image by W of every one of the hyperplanes η_s for $s \in S$ has length at least α. This will finish the proof of the lemma.

Let

$$\ell_i = w_i^{-1}[x_i, x_{i+1}] \subset w_i^{-1}\sigma_i = \sigma$$

Putting these line segments together gives a polygonal (that is, piecewise straight-line) path γ inside σ which touches each of the $n+1$ facets of σ. Then there is an elementary lower bound α for the length of γ, essentially given by the smallest 'altitude' of σ. ♣

Corollary: The set H of hyperplanes of the form $w\eta_s$ is *locally finite* in the affine space E.

Proof: In discussion of affine Tits' cones, we showed that compact subsets of $U - \{0\}$ meet only finitely-many hyperplanes of the form $w\eta_s$. Thus, the same property certainly holds for

$$E = E \cap U = E \cap (U - \{0\})$$

This is the desired local finiteness. ♣

Corollary: The group W is an *affine reflection group* generated by the reflections S in the hyperplanes $\eta_s \subset E$. Fixing $x_s \in \eta_s$, the n-simplex

$$C = \{x \in E : \langle x - x_s, e_s \rangle > 0\}$$

is a *chamber cut out by* H in E. ♣

Corollary: The chambers cut out by H all have compact closure. ♣

So what we have proven is, in part, that the disjoint pieces wF_I of the Tits' cone U yield a partition of E:

$$E = \bigsqcup_{w,I} w(F_I \cap E)$$

and that the sets $wF_I \cap E$ are the chambers cut out by the hyperplanes in E.

Consider the analogous partial ordering $wF_I \geq w'F_J$ if wF_I contains $w'F_J$ in its closure, restricting our attention to $i \neq S$. As noted in our earlier general discussion of the Tits' cone, the collection of sets wF_I with this partial ordering is isomorphic as poset to the 'abstract' Coxeter complex $\Sigma(W,S)$ attached to (W,S). The vertex set is the set of sets wF_{S-s_o}, that is, where the subset $I = S - s_o$ has cardinality just one less than S.

As in our discussion of geometric realizations (13.5), consider the map i from vertices of the Coxeter complex $\Sigma(W,S)$ to E given by

$$i(wW_{S-s_o}) = wF_{S-s_o} \cap E$$

By our lemma and its corollaries, the set of images $|i|(|x|)$ for simplices $x \in \Sigma(W,S)$ is *locally finite* in E, so we conclude that

Corollary: The map

$$|i| : |\Sigma(W,S)| \to E$$

of the geometric realization of the Coxeter complex to the affine space E is a *homeomorphism*. ♣

Remarks: And we will continue to use the fact that the group W acts as an affine reflection group, and cuts out a chamber which has finite diameter, as observed above.

13.7 The canonical metric

Beyond the fact that it is *possible* to put a metric on an affine Coxeter complex which makes it look like a Euclidean space, it is necessary to understand the metric aspects of *simplicial complex* automorphisms of these chamber complexes, and to *normalize* the metric. This is a preamble to the concommitant discussion for *buildings*.

Let $f : M_1 \to M_2$ be a map of metric spaces, where the metrics on M_i is $d_i(,)$. Say that f is a *similitude* if there is a constant λ so that for all $x, y \in M_1$

$$d_2(f(x), f(y)) = \lambda d_1(x,y)$$

Recall that, for an affine Coxeter system (W,S), just above we demonstrated a homeomorphism $|i|$ of the geometric realization $|\Sigma(W,S)|$ to a certain affine hyperplane E in the dual space V^* of the vector space V upon which we have the canonical linear representation. And we gave a *metric* on E so that W acts by *affine isometries*, and the chambers cut out by W have compact closure, so are of finite diameter. *Via* $|i|$, *define a* W*-invariant metric on* $|\Sigma(W,S)|$.

Keep in mind that by our definition if a system (W,S) is affine then it is *indecomposable*.

Theorem: Let $(W,S),(W',S')$ be affine Coxeter systems with metrics as just described. Let

$$\phi : \Sigma(W,S) \to \Sigma(W',S')$$

be an isomorphism of simplicial complexes. Then the geometric realization $|\phi|$ is a *similitude*

$$|\phi| : |\Sigma(W,S)| \to |\Sigma(W',S')|$$

Proof: We identify the geometric realizations of the Coxeter complexes with the affine spaces E,E' upon which W,W' act as affine reflection groups. Let $\langle,\rangle,\langle,\rangle'$ be the inner products on E,E', depending upon choice of base point. The groups W,W' preserve $\langle,\rangle,\langle,\rangle'$, respectively.

Fix the chamber C in E with facets $F_o,F_1,\ldots,F_n$ described by hyperplanes

$$\eta_i = \{x \in E_1 : \langle x,e_i\rangle = 0\} \ \text{ for } \quad i \geq 1$$
$$\eta_o = \{x \in E_1 : \langle x - x_o, e_o\rangle = 0\}$$

for arbitrary fixed $e_o \in \eta_o$. Here we take the e_i to be inward-pointing unit vectors orthogonal to η_i. By changing everything by a *dilation* of E we can suppose without loss of generality that

$$\langle x_o, e_o\rangle = 1$$

We can rewrite the defining condition for the 0^{th} facet as

$$\langle x, e_o\rangle = -1$$

Note that every dilation is a *similitude*.

Let $C' = \phi(C)$, and let $F_i' = \phi(F_i)$. Let the corresponding items for $(W',S'),C',F_o',\ldots,F_n'$ be denoted by the same symbols as for (W,S) and C but with primes.

Just above we observed that the Coxeter data can be recovered from the 'geometry' of the Coxeter complex. In particular, the number $m(s,t)$ (if finite) is half the cardinality of the set of chambers with face $W_{\{s,t\}}$. Thus, the two Coxeter matrices must be the same. Therefore,

$$\langle e_i, e_j\rangle = \langle e_i', e_j'\rangle'$$

since the Coxeter matrix determines these inner products.

Let $\Phi : E \to E'$ be the linear map defined by $\Phi(e_i) = e_i'$ *just for* $1 \leq i \leq n$. Then Φ preserves inner products, and $\Phi(e_o) = e_o'$ since for all i we have

$$\langle e_o, e_i\rangle = \langle e_o', e_i'\rangle'$$

Then also $\Phi(\eta_i) = \eta'_i$ since these hyperplanes are defined via the e_i, and the 0^{th} has been normalized by a dilation to be $\langle x, e_o\rangle = -1$. Thus, the orthogonal reflections through these hyperplanes are related by

$$\Phi s_i \Phi^{-1} = s'_i$$

Then also

$$\Phi W \Phi^{-1} = W'$$

That is, we have an isomorphism Φ_* of simplicial complexes, with

$$\Phi = |\Phi_*|$$

That is, the map Φ is the geometric realization of a simplicial complex map.

Since both ϕ and the simplicial complex map Φ_* take C to C' and take each F_i to F'_i, the *Uniqueness Lemma* (3.2) from our discussion of thin chamber complexes implies that $\Phi_* = \phi$. Thus, the 'geometric realization' $\Phi = |\phi|$ of $\Phi_* = \phi$ is an isometry. Of course, we had changed the original metrics on E and E' by similitudes. ♣

Corollary: Let $(W, S), (W', S')$ be affine Coxeter systems. Let

$$\phi : \Sigma(W, S) \to \Sigma(W', S')$$

be an isomorphism of simplicial complexes. Normalize the metrics on the geometric realizations $|\Sigma(W, S), |\Sigma(W', S')|$ by dilating so that the *diameter of a chamber is* 1 in both cases. Then

$$|\phi| : |\Sigma(W, S)| \to |\Sigma(W', S')|$$

is an *isometry* of the geometric realizations.

Proof: Note that we must know that the chambers are of finite diameter in order to normalize the metric so that the diameter is 1. Fortunately, we had proven earlier (13.6) that the chambers are ('geometric') n-simplices for $|\Sigma(W, S)|$ n-dimensional. Then the assertion follows from the proposition. ♣

We say that the metric normalized to give a chamber diameter 1, as mentioned in the previous corollary, is the **canonical metric** on the affine Coxeter complex $|\Sigma(W, S)|$.

13.8 The seven infinite families

We can illustrate the criteria for spherical-ness (13.2) and affine-ness (13.4), (13.6) of Coxeter complexes by the families $A_n, C_n, D_n, \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n$ described earlier in (2.1) and (2.2). Indeed, now we can substantiate the earlier description of the first three as spherical and the last four as affine.

In this discussion we will often refer to removal of a *vertex* from the Coxeter *diagram*, as being equivalent to removal of a *generator* from a Coxeter *system*. A minor benefit of this is that some colloquial geometric adjectives can be applied to these diagrams. For example, indecomposability of a system is equivalent to connectedness of the diagram.

To prove that A_n, C_n, D_n are *spherical*, we must prove in each case that the Coxeter matrix is *positive definite*. In general, to prove that a symmetric real matrix is positive definite, one must check that all the determinants of *principal minors* are *positive* (Recall that a *principal* minor is a submatrix obtained by removal of some columns *and the corresponding rows*; that is, if the $i_1, \ldots, i_k^{\text{th}}$ *columns* are removed then also remove the $i_1, \ldots, i_k^{\text{th}}$ *rows*, rather than removing a more arbitrary batch of k rows).

Removal of a generator from a diagram of type A_n leaves either a diagram of type A_{n-1} or a disjoint union of diagrams of types A_k and A_{n-k-1}. Removal of a generator from a diagram of type C_n leaves either a diagram of type C_{n-1}, or a disjoint union of types A_p and C_{n-p-1}, or a disjoint union of diagrams of types A_{n-2} and A_1. And removal of a vertex from a diagram of type D_n leaves either type A_{n-1}, or type D_{n-1}, or a disjoint union of A_1, A_1, and A_{n-3}.

Thus, to prove positive definiteness of all these, it suffices to do an induction. Thus, it suffices simply to prove that the *determinants* of the Coxeter matrices of these three types are *positive*. This computation can be done by expansion by minors, and is omitted.

To prove that $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n$ are *affine*, since the diagrams are all *connected* we must show that the Coxeter matrices are positive semi-definite but not positive definite. (The connectedness of the diagrams is evident). To do this, it would suffice to see that every (proper) principal minor is *positive*, and that the determinant of the whole is *zero*. That is, in part we must check that the diagrams obtained by removal of at least one vertex are all spherical. Happily, not only is this the case, but in fact the spherical types obtained are just the A_n, C_n, D_n just discussed. One might draw *pictures* of what happens to the diagrams.

The only new computation necessary is computation of the determinants, to check that they are zero. This can be done by expanding by minors, and we omit it.

The case of $\tilde{A}_1$ is somewhat special, being the infinite dihedral group, and can be treated directly.

Removal of any generator from the system of type $\tilde{A}_n$ (with $n \geq 2$) leaves a system of type A_n, which we have seen is spherical. Thus, $\tilde{A}_n$ is *affine*.

There are three sorts of vertices in the system of type $\tilde{C}_n$. In terms of the Coxeter diagram, there are the two vertices at the *ends*, that is, generators which commute with all but one other generator. If either of these is removed, the system remaining is of type C_n, which we have proven to be spherical. Second, there are the two generators *adjacent to* the ends. Removal of either of these yields a disconnected diagram, which is the disjoint union of a type A_1 and type C_{n-1}, so is spherical although reducible. Third, if $n \geq 4$, there are the generators not adjacent to the ends of the diagram. Removal of these yields a disjoint union of diagrams C_p and C_{n-p} for $2 \leq p \leq n-2$, which are again both spherical. Thus, $\tilde{C}_n$ is *affine*.

In the system of type $\tilde{D}_n$ there are three types of generators. First, there are the four extreme generators, which commute with all but one of the other generators. Removal of any of these gives a system of spherical type D_n. Second, removal of either of the two generators adjacent to the extreme generators gives a diagram which is the disjoint union of two copies of A_1, together with a D_{n-1}. Last, removal of any other vertex yields a disjoint union of two spherical types D_p and D_{n-p}. Thus, $\tilde{D}_n$ is *affine*.

In the system of type $\tilde{B}_n$ there are five types of generators. First, at the end of the diagram with the branch (oriflamme) there are the two generators removal of either of which leaves a diagram of spherical type C_n. Second, removal of the generator adjacent to the latter end leaves a disjoint union of diagrams A_1, A_1, and C_{n-1}. Third, removal of the generator at the *other* end leaves a spherical D_n. Fourth, removal of the generator adjacent to the latter one leaves a disjoint union of A_1 and D_{n-1}. Last, removal of any *other* generator leaves a disjoint union of spherical C_p and D_{n-p}. Thus, $\tilde{B}_n$ is *affine*.

Thus, granting our earlier discussion of affine and linear reflection groups, together with the linear algebra surrounding the Perron-Frobenius lemma, verification that these important families of Coxeter systems really are affine is not so hard. It is unlikely that one could reliably visualize the geometric realization of Coxeter complexes well enough to directly perceive that a given complex had geometric realization which was a Euclidean space.

14. Affine Buildings

- Affine buildings, trees: definitions
- The canonical metric
- Negative curvature inequality
- Contractibility
- Completeness
- Bruhat-Tits fixed point theorem
- Conjugacy classes of maximal compact subgroups
- Special vertices, compact subgroups

The canonical metrics put onto an *affine Coxeter complexes* in the last section will be stuck together now, in a canonical way, to obtain a canonical metric on an *affine building*, that is, a building all of whose apartments are affine Coxeter complexes.

At the end of this part are the first truly non-trivial applications of the building-theory to a class of groups including important families of p-adic matrix groups.

14.1 Affine buildings, trees: definitions

In this subsection we define *affine buildings*, of which the one-dimensional ones are *trees*. Several critical features of affine Coxeter complexes are recalled to emphasize the facts of the situation.

Let X be a thick building with apartment system $\mathcal{A}$. We have seen that each apartment $A \in \mathcal{A}$ is a complex $\Sigma(W, S)$ attached to a Coxeter system (W, S). From the discussion (4.4) using *links*, the chamber-complex isomorphism class of an apartment is independent of choice of apartment system, and is the same for all apartments. In particular, the isomorphism class of (W, S) is completely determined by the building X.

We say that X is an **affine building** if each apartment is an affine Coxeter complex. Emphatically, this requires that the Coxeter data be *indecomposable*, that is, that the Coxeter diagram be *connected*.

Recall that the requirement of *indecomposability* is that the generators S in (W, S) *cannot* be grouped into two non-empty disjoint sets S_1, S_2 so that $S = S_1 \cup S_2$ and so that $m(s_1, s_2) = 2$ for all $s_1 \in S_1$ and $s_2 \in S_2$. In effect, this requires that (W, S) not be a *product*. This requirement of indecomposability is not strictly necessary, but without it there are pointless complications.

Again, the *affineness* of the Coxeter matrix of (W, S) is the requirement that the Coxeter matrix be positive semi-definite, but not positive definite. (Already the indecomposability is used here to give such a simple criterion for affineness via the Perron-Frobenius lemma (13.3)).

It has been shown (13.6) that with these hypotheses the geometric realization $|\Sigma(W,S)|$ of the Coxeter complex $\Sigma(W,S)$ *is* an affine space in which W acts by affine reflection. And the *alcoves* or *chambers* cut out by all the reflecting hyperplanes are (literal) *simplices*.

The geometric realizations of these affine Coxeter complexes have *canonical metrics* (13.7), normalized so that the diameters of chambers are 1.

A **tree** is a *one-dimensional* thick affine building. That is, all the apartments are one-dimensional simplicial complexes. Then the geometric realizations of the apartments in a tree are isometric to the real line. (For us trees will play no special role).

The 'B' in the BN-pair attached to a group acting *strongly transitively* on an affine building is often called an **Iwahori subgroup**.

14.2 Canonical metrics on affine buildings

Here we establish only the crudest properties of the metrics which can be put on affine buildings. The more delicate *completeness* and *negative curvature inequality* will be established later, in preparation for the Bruhat-Tits fixed point theorem and its corollaries concerning maximal compact subgroups.

Let X be a thick affine building with apartment system $\mathcal{A}$. Recall that this includes the implicit hypothesis of *indecomposability* of the Coxeter system associated to the apartments, or, equivalently, *connectedness* of the Coxeter diagram.

In discussion of affine Coxeter complexes (13.7), it was proven that a simplicial complex isomorphism

$$\phi : \Sigma(W,S) \to \Sigma(W',S')$$

of (*indecomposable*) affine Coxeter complexes has geometric realization $|\phi|$ which is a *similitude*

$$|\phi| : |\Sigma(W,S)| \to |\Sigma(W',S')|$$

Further, if the metrics on the Coxeter complexes are normalized so that chambers have diameter 1, then $|\phi|$ is an *isometry*.

For $A \in \mathcal{A}$ let $|A|$ be the geometric realization (13.5) of A, *with the canonical metric*. The inclusions of simplicial complexes $A \to X$ give continuous inclusions of topological spaces $|A| \to |X|$.

Given $x, y \in |X|$, choose any apartment A so that $|A|$ contains both x and y, and define the **canonical metric** d_X on $|X|$ by

$$d_X(x,y) = d_A(x,y)$$

where $d_A(,)$ is the canonical metric (13.7) on A.

Proposition: The canonical metric on $|X|$ is well-defined.

Proof: Suppose that A, A' are two apartments both whose geometric realizations contain the points x and y. Then by one of the building axioms (4.1), there is a simplicial complex map $f : A \to A'$ which is the identity on $A \cap A'$. The fact from (13.7) mentioned above shows that this f must give rise to an *isometry* $|f| : |A| \to |A'|$ between the affine spaces $|A|$ and $|A'|$. Since f is the identity map on $A \cap A'$, the geometrically realized map $|f|$ is the identity map on the geometric realization $|A \cap A'|$. Since the points x, y both lie in $|A \cap A'|$, we see that

$$d_A(x, y) = d_{A'}(x, y)$$

That is, the distance between two points is independent of the choice of apartment containing them. ♣

For a chamber C, by abuse of notation write $|C|$ for the geometric realization of the simplicial complex consisting of C *and all its faces*.

In our discussion (4.2) of elementary properties of buildings, we considered the *retraction*

$$\rho = \rho_{A,C} : X \to A$$

centered at a chamber C of the apartment A. This is a simplicial complex map $X \to A$ which is the identity on A (hence, is a retraction). The Uniqueness Lemma (3.2) from the discussion of chamber complexes showed that there is at most one such retraction. Existence was a little more complicated to verify, but was a straightforward application of the axioms (4.1) for a building.

Theorem: Let X be an affine building with 'metric' $d = d_X$. Then

- The (geometric realizations of the) canonical retractions $\rho = \rho_{A,C} : X \to A$ centered at a chamber C in an apartment A do not increase 'distance', That is,

  $$d(|\rho|x, |\rho|y) \leq d(x, y)$$

  As a special case, if either x or y lies in $|C|$ then

  $$d(|\rho|x, |\rho|y) = d(x, y)$$

- The function d satisfies the triangle inequality, so really is a metric.
- For $x, y \in |X|$, and for any apartment A so that $|A|$ contains both x and y, let $[x, y]$ be the straight line segment connecting them, in the Euclidean geometry on $|A|$. Then the subset $[x, y]$ of $|X|$ does not depend upon A, and has the intrinsic characterization

  $$[x, y] = \{z \in |X| : d(x, y) = d(x, z) + d(z, y)\}$$

Proof: Again, for any other apartment A' *containing* C the restriction of ρ to a function $\rho : A' \to A$ is a simplicial complex isomorphism, by invocation of the Uniqueness Lemma. Thus, the proposition above shows that the geometrically realized map $|A'| \to |A|$ is an isometry.

So if $x \in |C|$, for any other $y \in |X|$ take an apartment A' containing C and so that $y \in |A'|$. Then we have the equality

$$d(|\rho|(x), |\rho|(y)) = d(x, y)$$

This is the special case of the first assertion.

And, for *any* chamber D in X, since by the axioms there is an apartment A' containing both C and D, the restriction

$$|\rho| : |D| \to |C|$$

is an isometry.

Given $x, y \in |X|$, let A' be an apartment so that $|A'|$ contains them both. By the local finiteness of the set of hyperplanes cutting out the chambers (12.1), (12.4), the line segment $[x, y]$ inside $|A'|$ connecting the two points meets $|D|$ for only finitely-many chambers D. Then we can subdivide the interval into pieces

$$[x, x_1] = [x_o, x_1], \ \ldots\ , [x_{n-1}, x_n] = [x_{n-1}, y]$$

so that each subinterval lies inside the geometric realization of a chamber inside $|A'|$. Then using the triangle inequality inside $|A|$ and the fact that $|\rho|$ is an isometry on chambers, we have

$$d(|\rho|x, |\rho|y) \le \sum d(|\rho|x_i, |\rho|x_{i+1}) =$$
$$= \sum d(x_i, x_{i+1}) = d(x, y)$$

This gives the general version of the first assertion, that $|\rho|$ is *distance-decreasing* (or anyway *non-increasing*).

To show that d satisfies the triangle inequality take x, y, z in $|X|$, let A be an apartment so that $|A|$ contains x, y, let C be a chamber in A, and let ρ be the retraction of X to A centered at C (4.2). Using the distance decreasing property of $|\rho|$ just proven, we have

$$d(x, y) \le d(x, |\rho|z) + d(|\rho|z, y) \le d(x, z) + d(z, y)$$

Thus we have the triangle inequality, as desired.

If we have equality

$$d(x, y) = d(x, z) + d(z, y)$$

then the inequalities in the previous paragraph must be equalities. From Euclidean geometry we find that $|\rho|z$ lies on the straight line segment $[x, y]$ connecting x and y. And to achieve the equalities above we must have

$$d(x, |\rho|z) = d(x, z) \qquad d(|\rho|z, y) = d(z, y)$$

so we must have

$$|\rho|z = tx + (1-t)y$$

with

$$t = d(z, y)/d(x, y)$$

Now this holds for *all* chambers C in A, so take C so that $|\rho|z$ lies in $|C|$. Then, from the special case of the first assertion of the theorem,

$$d(z, |\rho|z) = d(|\rho|(z), |\rho|z) = 0$$

From this,

$$|\rho|z = z$$

as desired. Thus, the line segment $[x, y]$ joining x, y has the indicated intrinsic characterization in terms of the metric. ♣

14.3 Negative curvature inequality

More properly, we will prove an inequality which could be construed as asserting that *affine buildings have non-positive curvature*. From this it will follow an affine building is *contractible*, and *complete* with respect to its canonical metric (14.2).

Let X be a thick *affine* building (14.1) with apartment system $\mathcal{A}$. That is, each apartment is an *affine* Coxeter complex $\Sigma(W, S)$. That is, the Coxeter matrix of (W, S) is indecomposable, positive semi-definite, but not positive definite. We have shown that the geometric realization $|\Sigma(W, S)|$ is an affine space.

For a chamber C, write $|C|$ for the geometric realization (13.5) of the simplicial complex consisting of C *and all its faces*.

Proposition: *Negative Curvature Inequality* Let X be an affine building with canonical metric $d = d_X$. For $x, y, z \in |X|$, let

$$z_t = tx + (1-t)y$$

Then

$$d^2(z, z_t) \leq td^2(z, x) + (1-t)d^2(z, y) - t(1-t)d^2(x, y)$$

Proof: First we recall that the construction of this point z_t makes sense and determines a unique point. Indeed, in proving the basic properties of the metric (14.2), we saw that the point

$$z_t = tx + (1-t)y$$

is indeed defined intrinsically, without reference to an apartment, as being the unique point q in $|X|$ so that

$$d(q, x) = td(x, y) \quad \text{and} \quad d(q, y) = (1-t)d(x, y)$$

More generally, we saw that the line segment $[x, y]$ is likewise defined independently of choice of apartment A so that $|A|$ contains both x and y.

Let A be an apartment so that $|A|$ contains x, y, and hence contains the line segment $[x, y]$. Fix another point $q \in |A|$. Let $E = |A|$ be Euclidean space *taking* q *as origin* with inner product $\langle , \rangle$ and associated norm $| * |$. The choice of q as origin is a minor cleverness which makes this computation much less ponderous.

We recall the simple identity

$$|x-y|^2 = |x|^2 - 2\langle x,y\rangle + |y|^2$$

From this we obtain

$$2\langle x,y\rangle = |x|^2 + |y|^2 - |x-y|^2 = d^2(q,x) + d^2(q,y) - d^2(x,y)$$

This allows us to compute

$$d^2(q,z_t) = |0-z_t|^2 = |tx+(1-t)y|^2 =$$
$$= t^2|x|^2 + 2t(1-t)\langle x,y\rangle + (1-t)^2|y|^2 =$$
$$= t^2d^2(q,x) + t(1-t)[d^2(q,x)+d^2(q,y)-d^2(x,y)] + (1-t)^2d^2(q,y) =$$
$$= td^2(q,x) + (1-t)d^2(q,y) - t(1-t)d^2(x,y)$$

In summary, for x,y,q all in the same apartment, we have an equality

$$d^2(q,z_t) = td^2(q,x) + (1-t)d^2(q,y) - t(1-t)d^2(x,y)$$

in place of the analogous inequality asserted in the proposition.

Now consider arbitrary $z \in |X|$. With a chamber D of $|A|$ so that $|D|$ contains z_t, let ρ be the retraction to A centered at D (4.2). Applying the previous identity to x,y,q with $q = |\rho|z$, we have

$$d^2(|\rho|z,z_t) = td^2(|\rho|z,x) + (1-t)d^2(|\rho|z,y) - t(1-t)d^2(x,y)$$

By the special version of the first assertion of the theorem,

$$d^2(z,z_t) = d^2(|\rho|z,z_t) =$$
$$= td^2(|\rho|z,x) + (1-t)d^2(|\rho|z,y) - t(1-t)d^2(x,y) \leq$$
$$\leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y)$$

where the last inequality follows from the general version of the distance-decreasing assertion. That is, we have the *comparison*

$$d^2(z,z_t) \leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y)$$

as asserted. ♣

14.4 Contractibility

Corollary: Let X be an affine building with canonical metric $d = d_X$. For $0 \leq t \leq 1$, let

$$z_t = tx + (1-t)y \in [x,y]$$

be the indicated affine combination of x and y. The function

$$t \times x \times y \to z_t = tx + (1-t)y$$

is a continuous function

$$[0,1] \times |X| \times |X| \to |X|$$

and $|X|$ is *contractible*.

Proof: First we prove continuity of

$$t \times x \times y \to z_t = tx + (1-t)y$$

Take t', x', y' close to t, x, y, respectively, let

$$z = t'x' + (1-t')y'$$

and apply the negative curvature inequality (14.3) to x', y', z in place of x, y, z. By continuity of the distance function, $d(z, x)$ is close to

$$d(z, x') = |t'x' + (1-t')y' - x'| = (1-t')|x' - y'| = (1-t')d(x', y')$$

and $d(z, y)$ is close to

$$d(z, y') = |t'x' + (1-t')y' - y'| = t'|x' - y'| = t'd(x', y')$$

Therefore, as t', x', y' go to t, x, y, we have

$$t'd^2(z, x') \to t(1-t)^2 d^2(z, x)$$

$$(1-t')d^2(z, y') \to t^2(1-t)d^2(z, y)$$

and trivially

$$t'(1-t')d^2(x', y') \to t(1-t)d^2(x, y)$$

Thus, the right-hand side of the curvature inequality goes to

$$t(1-t)^2 d^2(z, x) + t^2(1-t)d^2(z, y) - t(1-t)d^2(x, y) = 0$$

That is,

$$d^2(t'x' + (1-t')y', tx + (1-t)y) \to 0$$

This is the desired continuity assertion.

Taking y to be fixed in $|X|$ and considering the functions

$$f_t(x) = tx + (1-t)y$$

gives us

$$f_1 = \text{ identity map on } |X|$$

while

$$f_0(|X|) = \{y\}$$

which gives the desired *contraction* of $|X|$ to a single point.

14.5 Completeness

Now we prove completeness of an affine building. A fixed-point theorem would not be possible without this.

Theorem: The geometric realization $|X|$ of an affine building X, with its canonical metric, is *complete*.

Proof: Let ρ now be the *'labeling'* retraction $\rho : X \to \overline{C}$ of X to the complex $\overline{C}$ consisting of all faces of a given chamber C (4.4).

(Recall that we constructed this ρ by constructing a retraction ρ_A to C of each apartment A containing C, and then showing that these retractions had to agree on overlaps (from the building axioms (4.1)). The retractions ρ_A were constructed by iterating the map

$$f = f_{s_n} \circ f_{s_{n-1}} \circ \ldots \circ f_{s_2} \circ f_{s_1}$$

where f_s is the *folding* (3.3) of the thin chamber complex A along the s^{th} facet F_s of C, sending C to itself, and where $F_{s_1}, \ldots, F_{s_n}$ are all the facets of C. For example, this folding sends the chamber sC of A to C, where sC is the unique chamber in A with facet F_s.)

As with the retractions to apartments (4.2) considered above in proving the negative curvature inequality (14.3), the geometric realization of this ρ, when restricted to $|D|$ for any chamber D, is an isometry, and is altogether distance-decreasing. The only new ingredients needed to prove this are the observations that the action of the associated Coxeter group W on the apartment $\Sigma = \Sigma(W, S)$ is by isometries, is transitive on chambers, and is type-preserving.

Therefore, given a Cauchy sequence $\{x_i\}$ in $|X|$, the image $\{|\rho|x_i\}$ is a Cauchy sequence in $|C|$. Since $|C|$ is a closed subset of a complete metric space it is complete, so $\{|\rho|x_i\}$ has a limit y.

For each x_i let C_i be a chamber in X so that $x_i \in |C_i|$, and let y_i be the unique point in $|C_i|$ so that $|\rho|y_i = y$. Since $|\rho|$ restricted to $|C_i|$ is an isometry,

$$d(x_i, y_i) = d(|\rho|x_i, y) \to 0$$

Therefore, since $\{x_i\}$ is Cauchy, it must be that $\{y_i\}$ is Cauchy.

Lemma: The inverse image in $|X|$ by $|\rho|$ of a single point y of $|C|$ is *discrete* in $|X|$.

Proof: Generally, given x in the geometric realization $|Y|$ of a simplicial complex Y, let the **star of x in Y** be the union $\mathrm{st}(x)$ of the geometric realizations $|\sigma|$ for simplices $\sigma \in Y$ so that $x \in |\sigma|$.

We *claim* that there is $\delta > 0$ so that for all $x \in |X|$ with $|\rho|x = y$ the star of x in X contains the ball of radius δ in $|X|$ with center at x. It is immediate that this star contains no other point x' also mapping to y by $|\rho|$, so for another point x' mapping to y we have

$$d(x, x') \geq \delta$$

This would give the desired discreteness property.

To prove the claim: take any apartment A containing C, and let H be the locally finite collection of reflecting hyperplanes associated to the affine reflection (Coxeter) group W acting on $|A|$ (12.1), (12.4), (13.4). Let δ be the infimum of the distances from the point y to hyperplanes not containing it. The local finiteness assures that this infimum is *positive*. Thus, for $z \in |A|$ with $d(y, z) < \delta$ the line segment $[y, z]$ does not *cross* any hyperplane (although it may lie entirely inside one or more). Thus, in the Tits' cone notation, the *open* line segment (y, z) lies inside some face F_I. Therefore, both y and z lie in the topological closure of F_I. Therefore, F_I is a subset of the star of y in A, and z lies inside the star of y in A.

More generally, if $d(z,x) < \delta$ and $|\rho|x = y$, there is an apartment A' so that $|A'|$ contains both x and z. There is a simplicial complex isomorphism $\phi : A' \to A$ so that the $|\phi|x = y$; we have seen that $|\phi|$ must be an isometry. Then

$$d(y, |\phi|z) = d(|\phi|x, |\phi|z) = d(x,z) < \delta$$

By the previous paragraph, $|\phi|z$ must lie in the star of y in A. Therefore, since ϕ was a simplicial complex isomorphism, z had to be in the star of x in A'. This is certainly a subset of the star of x in all of X. Thus, the star of x in X contains the ball of radius $\delta > 0$ around x, as desired. ♣

By this lemma, returning to the proof of the last assertion of the theorem, we see that the Cauchy sequence $\{y_i\}$ must eventually be constant, equal to some z with $|\rho|z = y$. Since $d(x_i, y_i) \to 0$, it must be that $x_i \to z$. This completes the proof. ♣

14.6 Bruhat-Tits fixed-point theorem

We will invoke only a special case of the negative curvature inequality (14.3), with $t = \frac{1}{2}$ (in the notation there). And we abstract it a little.

Specifically, we suppose that we have a complete metric space M with metric d so that, given $x, y \in M$ there is a point $m \in M$ so that for all $z \in M$

$$d(z,m)^2 \leq \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2$$

(In the case of an affine building the point m was the midpoint of the line segment connecting x, y). An *isometry* of a metric space is simply a map $\phi : M \to M$ so that

$$d(\phi(x), \phi(y)) = d(x,y)$$

for all $x, y \in M$.

Theorem: Let G be a group of isometries of the complete metric space (M,d). If there is a non-empty, bounded, G-stable subset of M, then G has a fixed point on M.

Proof: Let Y be a non-empty bounded subset of M. For $x \in M$, let

$$r_x(Y) = \sup_{y \in Y} d(x,y)$$

The **circumradius** of Y is

$$r(Y) = \inf_{x \in M} r_x(Y)$$

If $x \in X$ is such that $r_x(Y) = r(Y)$, then x is a **circumcenter** of Y.

Clearly if f is an isometry of M and if x is a circumcenter of a set Y, then $f(x)$ is a circumcenter of $f(Y)$, since the notion of circumcenter is respected by distance-preserving maps. Thus, the collection of circumcenters of a G-stable set must be G-stable. Therefore, we will be done if we show that every non-empty bounded subset Y of M has a *unique* circumcenter.

With $z \in Y$ we have

$$r_m(Y)^2 \leq \frac{1}{2} r_x(Y)^2 + \frac{1}{2} r_y(Y)^2 - \frac{1}{4} d(x,y)^2$$

where the point m is as above, given $x, y \in M$. By rearranging,

$$d(x,y)^2 \leq 2r_x(Y)^2 + 2r_y(Y)^2 - 4r_m(Y)^2 \leq 2r_x(Y)^2 + 2r_y(Y)^2 - 4r(Y)^2$$

since certainly $r(Y) \leq r_m(Y)$. If both x and y were circumcenters, then the right-hand side would be zero, so $x = y$. This is the *uniqueness* of the circumcenter.

On the other hand, if we had a sequence of points x_n so that $r_{x_n}(Y) \to r(Y)$, then the last inequality applied to x_i, x_j in place of x, y gives

$$d(x_i, x_j)^2 \leq 2r_{x_i}(Y)^2 + 2r_{x_j}(Y)^2 - 4r(Y)^2$$

The right-hand side goes to zero as the infimum of i, j goes to ∞, so $\{x_i\}$ is necessarily a Cauchy sequence in M. The completeness of M assures that this Cauchy sequence has a limit, which evidently is the circumcenter. This proves *existence*. ♣

14.7 Maximal compact subgroups

The main purpose is to classify conjugacy classes of maximal compact subgroups of groups G acting on affine buildings (14.1). Actually, rather than *compact subgroup*, the weaker and more general notion of *bounded subgroup* is appropriate. This is defined just below.

The first result we give determines conjugacy classes of maximal bounded subgroups in a group with a *strict* affine BN-pair obtained from an appropriate action on a thick building (5.2). Here the group is required to act strongly transitively and preserve types on a thick *affine* building. This is a cleaner result than the more general second result, for a *generalized* affine BN-pair (5.5).

At the outset we 'recall' the standard nomenclature for discussion of *bounded* sets in a manner not depending upon a metric nor upon compactness.

A **bornology** on a set G is a set $\mathcal{B}$ of subsets of G, called the *bounded subsets* of G, so that

- Every singleton set $\{x\}$ is in $\mathcal{B}$.
- If $F \subset E$ and $E \in \mathcal{B}$ then $F \in \mathcal{B}$.
- A finite union of elements of $\mathcal{B}$ is again in $\mathcal{B}$.

Suppose further that G is a *group*. It is a **bornological group** if, in addition to the previous requirements, we have

- For $E, F \in \mathcal{B}$ the set $EF = \{ef : e \in E, f \in F\}$ is in $\mathcal{B}$.

- If $E \in \mathcal{B}$ then $E^{-1} = \{e^{-1} : e \in E\}$ is in $\mathcal{B}$.

Let X be a *thick affine building* (14.1). Let $\tilde{G}$ be a group acting upon X by simplicial complex automorphisms, and suppose that the subgroup G of $\tilde{G}$ consisting of *type-preserving* elements is *strongly transitive*.

Inside G we have a *strict* BN-pair (5.2): let B be the stabilizer in G of a chamber C, and let $\mathcal{N}$ be the stabilizer in G of an apartment A containing C. The pair $(B, \mathcal{N})$ is a (strict) **affine BN-pair** in G. Put $T = B \cap \mathcal{N}$. Then $W = \mathcal{N}/T$ is the associated Coxeter group, with generators S given by reflections in the facets of C.

Let $\tilde{\mathcal{N}}$ be the stabilizer in $\tilde{G}$ of C, let $\tilde{B}$ be the stabilizer in $\tilde{G}$ of A, and let $\tilde{T} = \tilde{\mathcal{N}} \cap \tilde{B}$ be the intersection. The general discussion (5.5) of generalized BN-pairs showed that $\Omega = \tilde{T}/T$ is *finite*, and that $\tilde{G} = G\tilde{T}$, for example.

(Emphatically, the assumption of affineness is that the associated Coxeter complex $\Sigma(W, S)$ is *affine*, and that this implicitly includes a hypothesis of *indecomposability*, that is, connectedness of the Coxeter diagram).

Define a bornology $\mathcal{B}$ on G by saying that $E \in \mathcal{B}$ if and only if E is contained in a finite union of double cosets BwB.

The elementary facts about the Bruhat-Tits decomposition, e.g., the cell multiplication rules (5.1), show that this set $\mathcal{B}$ is indeed a bornology on G, so making G a bornological group.

Remarks: If the group G has a topology in which B is in fact *compact and open*, then 'bounded' is equivalent to 'having compact closure'.

We give two theorems here, the first treating the simpler case of the *strict* BN-pair, the second treating the general case. As preparation we need the comparison of notions of boundedness given by the following proposition. We will need this again for the *generalized* BN-pair situation, so we give the general version of the proposition here.

Proposition: The following three conditions on a subset E of $\tilde{G}$ are equivalent:

- E is contained in a finite union of double cosets $B\sigma wB$ with $w \in W$ and $\sigma \in \Omega$.
- There is a point $x \in |X|$ so that $Ex = \{gx : g \in E\}$ is a bounded subset of the metric space $|X|$.
- For every bounded subset Y of the metric space $|X|$, the set $EY = \{gy : g \in E\}$ is bounded in $|X|$.

Remarks: Note that this applies as well to subsets of G, in which case elements $\sigma \in \Omega$ can be ignored.

Proof: To prove that the first condition implies the second, let $x \in |C|$ where C is the chamber fixed by B. Then for $g = b\sigma wb' \in B\sigma wB$,

$$d(x, gx) = d(x, b\sigma wb'x) = d(x, b\sigma wx) = d(b^{-1}x, \sigma wx) = d(x, \sigma wx)$$

since B fixes any $x \in |C|$ and since the whole group acts by isometries. Thus, $B\sigma wBx$ is contained in the closed ball of radius $d(x, \sigma wx)$ centered at x. From this, the first condition implies the second.

Now let Y be a bounded subset of $|X|$ and $x \in |X|$ a point so that Ex is bounded. In particular, let δ be a bound so that $d(x, gx) \leq \delta$ for all $g \in E$, and let D be a bound so that $d(x, y) \leq D$ for all $y \in Y$. Then, for $y \in Y$ and $g \in E$,

$$d(x, gy) \leq d(x, gx) + d(gx, gy) = d(x, gx) + d(x, y) \leq \delta + D$$

Thus, the second implies the third.

Assume that EY is bounded, where $Y = |C|$. Let A be the apartment containing C whose stabilizer is $\mathcal{N}$. Let $\rho : X \to A$ be the canonical (4.2) retraction of the whole building to A, centered at C. As discussed earlier (14.2), $|\rho|$ does not increase distances, so $|\rho|(E|C|)$ is a bounded subset of $|A|$. The set of reflecting affine hyperplanes in $|A|$ is locally finite (12.1), (12.4), (13.4), so a bounded subset meets only finitely-many chambers.

We have shown that

$$\Omega = \tilde{T}/T \approx \tilde{G}/G$$

is *finite*. Let Ξ be a choice of representatives in $\tilde{T}$. In our discussion of the Bruhat decomposition (5.2) we showed that an element g in the type-preserving subgroup G lies in BwB where $w \in W$ is such that $\rho(gC) = wC$. Thus, for $\tilde{g} = g\sigma_i \in \tilde{G}$ with $g \in G$ and $\sigma_i \in \Xi$, we have $\rho(\tilde{g}C) = wC$, since σ_i also stabilizes C. Since $|\rho|(E|C|)$ is contained in the geometric realizations of finitely-many chambers in A, certainly $\rho(EC)$ is a finite union of chambers. Thus, it follows that E is contained in finitely-many double cosets $Bw\Xi B$, and each such is a finite union of double cosets $B\sigma wB$.

This proves the proposition. ♣

Theorem: We assume that G acts strongly transitively and preserves types on a thick affine building X. With the bornology above, every bounded subgroup of G is contained in a *maximal* bounded subgroup. The maximal bounded subgroups of G are exactly the stabilizers of *vertices* of X. Each conjugacy class of maximal bounded subgroups contains a unique one from among the maximal bounded subgroups

$$K = \bigsqcup_{w \in W_{S'}} BwB$$

where $S' = S - \{s_o\}$ for some $s \in S$ and where $W_{S'} = \langle S' \rangle$ is the *special* subgroup of W generated by S'.

Remarks: Indeed, the stabilizer of the vertex of C of type $S' = S - \{s_o\}$ is the special subgroup

$$K = \bigsqcup_{w \in W_{S'}} BwB = BW_{S'}B$$

From the Perron-Frobenius lemma (13.3) and its application to Coxeter groups (13.6), the assumption of affine-ness assures that any group $W_{S'}$ with S' a *proper* subset of S is *finite*. Thus, such groups K really are *bounded* in the present sense.

We will prove this theorem along with the more general version given just below, which we state first.

Recall that in discussion (5.5) of *generalized* BN-pairs the following facts were verified. The groups $\mathcal{N}, B$ are normalized by $\tilde{T}$, and conjugation by elements of $\tilde{T}$ stabilizes S, as automorphisms of A. And the group G is a normal subgroup of $\tilde{G}$, of finite index, with $\tilde{G} = \tilde{T}G$. Let $\Omega = \tilde{T}/T$ as above. Then for $\sigma \in \Omega$ and $w \in W$, $\sigma B = B\sigma = B\sigma B$ and

$$\sigma BwB = B\sigma wB = B(\sigma w\sigma^{-1})B\sigma$$

where we note that $\sigma w \sigma^{-1} \in W$. In particular, from this we see that it is reasonable to take the bornology on $\tilde{G}$ in which the bounded subsets are those contained in finitely-many double cosets $B\sigma wB$, where $\sigma \in \Omega$ and $w \in W$.

Theorem: Let $\tilde{G}$ act strongly transitively on a thick affine building X, with type-preserving subgroup G acting strongly transitively. With the bornology above, every bounded subgroup of $\tilde{G}$ is contained in a *maximal* bounded subgroup. Every maximal bounded subgroup K of $\tilde{G}$ is the *stabilizer of a point in* X. Conjugating if necessary, we may assume that $B \subset K$. The subgroup $K_o = K \cap G$ is bounded in G and is of the form $BW_{S'}B$ for some *proper* subset S' of S. Identifying K/K_o with a subgroup Ω_K of $\Omega = \tilde{T}/T$, we have

$$K == \Omega_K K_o = \Omega_K \cdot BW_{S'}B = B\Omega_K W_{S'}B$$

Remarks: Note that it is *not* asserted that for *every* point y in $|X|$ the stabilizer of y is *maximal*, although the proposition above proves that it is *bounded*. And, unlike the previous theorem where the *points* mentioned here were always *vertices* in the simplicial complex, we no longer have any such simplicity.

Remarks: In this generality it is not clear which subgroups of Ω are candidates for appearance as Ω_K. For example, in general there is no reason to expect $\tilde{T}$ to be a bounded subgroup, so there is no reason to expect that the whole group Ω could appear as an Ω_K.

Proof: We prove both theorems at once, with two different endings to the proof.

Since $\tilde{G}$ acts on X, it acts on its geometric realization $|X|$. Our discussion of affine Coxeter complexes and affine buildings assures that the action on $|X|$ is by *isometries*. The *negative curvature inequality* assures that the hypotheses of the Bruhat-Tits Fixed-Point theorem are fulfilled. The proposition above relates the bornology on G or $\tilde{G}$ to the metric on $|X|$. In particular, it shows that the stabilizer of a point is indeed a bounded subgroup.

Conversely, given a bounded subgroup K of G, take any $x \in |X|$. Then K stabilizes the set Kx, which by the proposition is a bounded subset of $|X|$. Thus, K has a fixed point $x_o \in |X|$, by the Bruhat-Tits fixed-point theorem (14.6). Thus, K is surely contained in the fixer of the point x, which is maximal bounded.

Now let K be maximal bounded, fixing a point $x \in |X|$. Since G is transitive on chambers, by conjugation by G we can assume that x is in the closure of the fundamental chamber C (stabilized by B), so $B \subset K$.

The type-preserving property of G yields a simple conclusion in that case. Let τ be the smallest simplex τ in X so that $x_o \in |\tau|$. Since G is type-preserving, $g \in G$ stabilizes the geometric realization of a simplex if and only if it fixes all vertices of the simplex. Thus, the stabilizer K in G of x_o is the stabilizer of τ, which is the intersection of the stabilizers of the vertices of τ. That is, the *maximal* bounded subgroups are exactly the stabilizers of *vertices* in X. This proves the theorem for the type-preserving group G.

In particular, the bounded subgroup $K \cap G$ of $\tilde{G}$ must be of the form $BW_{S'}B$ for some subset S' of S. By the Perron-Frobenius theory, the subset S' must be a *proper* subset of S for $W_{S'}$ to be finite, since (W, S) is *affine* (which entails indecomposability).

Let $\Omega_K = K/K_o$, viewed as a subgroup of the finite group $\Omega = \tilde{T}/T$. Since K_o contains $B \supset T$ and $\tilde{T}G = \tilde{G}$, we can indeed choose representatives in $\tilde{T}$ modulo K_o for all elements of K. Then

$$K = \Omega_K K_o$$

This is the second theorem. ♣

14.8 Special vertices, good compact subgroups

Only *some* of the maximal compact (or maximal bounded) subgroups of a group acting on a thick affine building are suitable for subsequent applications. In this subsection we give a definition of *'good' maximal bounded subgroup*, and see that, as a corollary of the classification of maximal bounded subgroups there *is* at least one such, by relating *good* subgroups to *special* vertices.

The definition alone already requires our previous results (12.4) on *affine reflection groups*.

Let $\tilde{G}$ be a group acting on a thick affine building X (14.1). Let G be the subgroup of $\tilde{G}$ preserving types, and suppose that the group G acts strongly transitively on X. Let (W, S) be the Coxeter system attached to G: by hypothesis this system is *affine* (and, implicit in this is the assumption of indecomposability, that is, connectedness of the diagram).

Fix an apartment A. Let $|A|$ be its geometric realization (13.5) which we view as a real vectorspace equipped with an inner product, with respect to which the group W acts by *isometries*. (Recall (12.4) that W acts by affine maps on $|A|$). Let $w \to \bar{w}$ be the map which associates to an element w of W

its *linear part*. Let $\bar{W}$ be the group of all linear parts, which we have shown to be *finite* in our general discussion (12.4), (13.4) of affine reflection groups.

Fix a chamber C in A, and let B be the stabilizer in G of C.

A maximal bounded subgroup K of G containing B is **good** if it contains representatives for $\bar{W}$.

Remarks: To give a useful definition of *good* maximal bounded subgroup without reference to B and $\bar{W}$ is somewhat awkward, and serves no immediate purpose.

Corollary: There exist *good maximal bounded* subgroups of G, obtained as BW_xB where W_x is the fixer in W of a *special vertex* x of the chamber fixed by B.

Proof: From the fixed-point theorem corollaries of the previous section, the maximal bounded subgroups are exactly stabilizers BW_xB of vertices x, where W_x is the stabilizer in W of x.

For a special vertex x, the fixer of x in BW_xB contains representatives for W_x, which maps isomorphically to $\bar{W}$ (by definition of *special*). (And in discussion (12.4) of affine reflection groups it was proven that there always exist special vertices). ♣

For a vertex x of C, let S_x be the subset of S consisting of those reflections in S which fix x. That is, S_x consists of all reflections in S except the reflection through the facet of C opposite the vertex x.

Corollary: There exist *good bounded* subgroups of $\tilde{G}$, obtained as $\Omega' K_o$ where K_o is a good maximal bounded subgroup of G, and where Ω' is a *bounded* subgroup of $\tilde{T}$ stabilizing the subset S_x of S under the conjugation action of $\tilde{T}$. ♣

Remarks: While much of the interest here is in the subsequent study of *good maximal compact subgroups*, the substance of the result resides in the fact that *special vertices* exist in thick affine buildings. And then the fixed-point theorem together with general facts about Bruhat-Tits decompositions entail existence of the *good* maximal compact subgroups.

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15. Finer Combinatorial Geometry

- Minimal galleries and reduced galleries
- Characterizing apartments
- Existence of prescribed galleries
- Configurations of three chambers
- Subsets of apartments, strong isometries

This section does *not* use the hypothesis of affine-ness. Rather, it is a relatively elementary but more refined discussion of buildings *in general*. It could have taken place earlier, but was not necessary for earlier use.

15.1 Minimal galleries and reduced galleries

Let X be a thick building with labeling λ (4.1), (4.4). Extending the notion discussed earlier (3.4), (3.5), (3.6) for for Coxeter complexes, the **type** of a *non-stuttering* gallery

$$\gamma = (C_o, C_1, \ldots, C_n)$$

is the list

$$\lambda(\gamma) = (\lambda(C_o \cap C_1), \lambda(C_1 \cap C_2), \ldots, \lambda(C_{n-1} \cap C_n))$$

of labels of the common facets of adjacent chambers.

Fix an apartment A_o in X, and fix a chamber C_o in A_o. Then (4.2) we may identify A_o with a Coxeter complex (W, S), and the generators S with reflections in the facets of the fixed chamber C_o in A_o. Further (4.4), we may take the label map λ to be a retraction of X to C_o, thereby allowing us to identify the labels of facets with elements of the generating set S of W.

Thus, we can view the *type* of a gallery $\gamma = (C_o, \ldots, C_n)$ as giving a *word* in the elements of S as follows: for

$$\lambda(C_{i-1} \cap C_i) = s_i \in S$$

we have

$$\lambda(\gamma) = (s_1, s_2, \ldots, s_n)$$

and we can consider the word

$$(s_1, \ldots, s_n)$$

thus associated to γ. Even though a *word* is merely a list of elements of the set S, we may often behave as though such a word were the product $s_1 \ldots s_n$ inside W rather than the n-tuple.

We say that γ is **reduced** if this *word* is reduced, in the sense that its length is what it appears to be, that is, if

$$\ell(s_1 \ldots s_n) = n$$

Proposition: Fix two chambers C_o, C_n in the thick building X. Let $\gamma = (C_o, \ldots, C_n)$ be a gallery connecting C_o to C_n. Then γ is *minimal* if and only if it is *reduced*.

Proof: Suppose that γ is minimal among galleries connecting C_o, C_n. Then γ lies in *every* apartment containing both these chambers, by the *combinatorial convexity of apartments* (4.5). Any such apartment A is a Coxeter complex $\Sigma(W,S)$. Then the labeling of a gallery corresponds to its description as

$$C_o, s_1C_o, s_1s_2C_o, s_1s_2s_3C_o, \ldots, s_1, \ldots, s_nC_o$$

where the s_i are in S. That is, the label is $(s_1, \ldots, s_n)$. In our earlier study of Coxeter complexes (3.4) we showed that such gallery *inside a Coxeter complex* is minimal if and only if the word $s_1 \ldots s_n$ is reduced, that is, has length n.

On the other hand, suppose that the type of a non-stuttering gallery $\gamma = (C_o, \ldots, C_n)$ is reduced. By induction, we may suppose that the gallery $(C_1, \ldots, C_n)$ is minimal, so lies inside an apartment A, by the combinatorial convexity of apartments (4.5). Let ρ be the canonical retraction of the building to A, centered at C_1 (4.2). Thus, $\rho(C_i) = C_i$ for $i \geq 1$. The image of γ under ρ is a gallery with the same type, since the retraction ρ preserves labels (4.4). The further crucial point is that $\rho(C_o) \neq C_1$, since ρ preserves lengths of minimal galleries from C_1 to other chambers in the building (4.2). Thus, $\rho(\gamma)$ is *non-stuttering* and reduced inside an apartment, so is minimal. Then γ itself must have been minimal. ♣

15.2 Characterizing apartments

Now we can give a geometric characterization of apartments in the *maximal* apartment system. We use the idea of *type* of a gallery, and the result of the previous section that *reduced type* is equivalent to *minimality* of a gallery.

Now let X be a thick building, with *maximal* apartment system $\mathcal{A}$. In the course of proving that there is a maximal apartment system (4.4) it was shown that there is a Coxeter system (W,S) so that every apartment in $\mathcal{A}$ is isomorphic to the Coxeter complex $\Sigma(W,S)$ attached to (W,S). And when two apartments A, A' have a chamber C in common, the isomorphism $A \to A'$ fixing C and its faces is unavoidably label-preserving (4.4).

Let σ be a subcomplex of X which is a chamber complex itself, and whose dimension is the same as that of X. (The last condition is that the dimension of a maximal simplex in σ is the dimension of a maximal simplex in X).

Theorem: The subcomplex σ is an apartment in the *maximal* system $\mathcal{A}$ if and only if σ is isomorphic to $\Sigma(W,S)$ by a simplicial complex map *preserving labels*.

Proof: The idea is to prove that adjoining σ to the maximal apartment system $\mathcal{A}$ still satisfies the axioms (4.1) for an apartment system, so σ must be in $\mathcal{A}$.

To prove the claim, we verify that

$$\mathcal{A}' = \mathcal{A} \cup \{\sigma\}$$

satisfies the axioms for apartment systems in a building:

Since each apartment $A \in \mathcal{A}$ is a thin chamber complex (actually a Coxeter complex), and since σ is such by hypothesis, then every complex in $\mathcal{A}'$ is certainly a thin chamber complex.

The condition that any two simplices lie in a common apartment is certainly met by $\mathcal{A}'$, since this already holds for $\mathcal{A}$.

The only axiom whose verification is non-trivial is the requirement that, given two complexes $x, y \in \mathcal{A}'$ with a common chamber C, there is a chamber-complex isomorphism $x \to y$ fixing every simplex in $x \cap y$. Certainly we need only consider the case that $x = \sigma$ and $y = A \in \mathcal{A}$.

By hypothesis, there is a label-preserving isomorphism $f : \sigma \to A$. Since the Coxeter group W of type-preserving automorphisms of $A \approx \Sigma(W, S)$ is transitive on chambers (3.4), we can adjust f so that $f(C) = C$. It is *not* yet clear that this f fixes $\sigma \cap A$.

On the other hand, let ρ be the retraction of X to A centered at C (4.2), and consider the restriction $\rho_o : \sigma \to A$ of ρ to σ. By definition of retraction (3.1), ρ_o fixes $\sigma \cap A$.

Thus, f and ρ_o agree on the chamber C, and map to the thin chamber complex A. Let γ be a minimal (necessarily non-stuttering) gallery in σ starting at C. The image $f(\gamma)$ is non-stuttering since f is an isomorphism. If we can prove that $\rho_o(\gamma)$ also must be non-stuttering, then by the *Uniqueness Lemma* (3.2), we could conclude that $f = \rho_o$, verifying the last axiom for a building and an apartment system.

Now $f(\gamma)$ is minimal in A, so (3.4), (3.6) it is of *reduced type*. Thus, since f is a type preserving isomorphism, γ itself is of reduced type. Thus (15.1), it is a minimal gallery *in the building*.

Thus, since the retraction ρ preserves the lengths of galleries starting at C, the length of $\rho(\gamma)$ must be the same as that of γ, so $\rho(\gamma)$ must be non-stuttering. That is, the restriction ρ_o of ρ to σ maps γ to a non-stuttering gallery.

This allows application of the Uniqueness Lemma (3.2), which yields $f = \rho_o$. That is, the postulated isomorphism f really is the identity on $\sigma \cap A$, since ρ is the identity on A, by definition. This verifies the requisite axiom. ♣

15.3 Existence of prescribed galleries

The development here uses a continuation of the idea of *type of a gallery* discussed just above. We define a sort of *Coxeter-group-valued distance function* δ on chambers in a thick building. Very roughly put, the main result in this section asserts that two chambers can be connected by galleries of all *plausible* types.

First, an observation: In a Coxeter complex $A = \Sigma(W, S)$ we can define a W-valued function δ on *pairs* of chambers of A by

$$\delta(\{w_1\}, \{w_2\}) = w_1^{-1}w_2$$

where we recall that the chambers in $\Sigma(W, S)$ are singleton subsets of W. Note that this is a refinement of the notion of length of minimal gallery, since here the length of the element $w_1^{-1}w_2 \in W$ is the length of any minimal gallery from $\{w_1\}$ to $\{w_2\}$.

Let (W, S) be the Coxeter system so that the apartments of X are Coxeter complexes $\Sigma(W, S)$. For two chambers C_o, C_n in X, let γ be a minimal (non-stuttering) gallery from C_o to C_n. As above (15.2), we define the *type* of γ as follows. We have proven (4.5) that such a minimal gallery lies inside some apartment A, which we view as identified with $\Sigma(W, S)$ (4.3). Then there is a sequence $s_1, s_2, \ldots, s_n$ of elements of S so that the gallery is

$$\gamma = (C_o, s_1C_o, s_1, s_2C_o, s_1s_2s_3C_o, \ldots, s_1 \ldots s_nC_n = C_n)$$

The type of γ is the *word*

$$(s_1, \ldots, s_n)$$

We define

$$\delta(C_o, C_n) = s_1 \ldots s_n \in W$$

That is, while the *type* of a gallery is not quite an *element* of the group W, but rather is just a *word* in the generators S, this function δ *does* take values in the group itself.

Lemma: The W-valued function δ on pairs of chambers in the thick building X really is well-defined.

Proof: We must show first that *any* identification of an apartment with the Coxeter complex $\Sigma(W, S)$ gives the same value for δ on two chambers inside A. Second, we must show that the value $\delta(C_o, C_n)$ does not depend on the choice of apartment A containing the two chambers.

It is not hard to see that two different identifications of an apartment with $\Sigma(W, S)$ differ by a label-preserving automorphism of $\Sigma(W, S)$. The group W is certainly transitive on chambers in $\Sigma(W, S)$, and the Uniqueness Lemma (3.2) shows that two label-preserving automorphisms which agree on a chamber must be identical. Thus, as we have observed on other occasions as well, W itself gives all the label-preserving automorphisms of $\Sigma(W,S)$. Thus, the simple computation

$$\delta(w_1, w_2) = w_1^{-1} w_2 = (ww_1)^{-1} (ww_2) = \delta(ww_1, ww_2)$$

shows that δ is well-defined on each apartment.

Now let A, B be two apartments both containing C_o, C_n. By the building axioms (4.1), there is an isomorphism $f : A \to B$, and we proved that f is unavoidably label-preserving. Thus, if we have a minimal gallery γ in A from C_o to C_n, its image $f(\gamma)$ in B is a minimal gallery *of the same type*. Thus, the value $\delta(C_o, C_n)$ does not depend upon which of the two apartments A, B we use to connect the two chambers by a gallery. ♣

Proposition: Fix a chamber C in an apartment A. For any other chamber D in the thick building X, we have

$$\delta(C, D) = \delta(C, \rho D)$$

where $\rho = \rho_{A,C}$ is the retraction of X to A, centered at C.

Proof: Let γ be a (non-stuttering) minimal gallery from C to D. The retraction ρ preserves the lengths of such galleries, and preserves types as well (4.2), (4.4). ♣

Theorem: Let C_o, C_n be two chambers and $\delta(C_o, C_n) = w \in W$. Then for *any* reduced expression

$$w = s_1 s_2 \ldots s_n$$

for w, there is a minimal gallery of type $(s_1, \ldots, s_n)$ connecting C_o to C_n. In fact, this can be accomplished inside any chosen apartment containing both chambers.

Proof: By the building axioms (4.1), the two chambers do lie in some common apartment A. Having seen that δ is well-defined, we may as well take $A = \Sigma(W,S)$, and, for that matter, $C_o = \{1\}$. Then $C_n = \{w\}$. By this point it is clear that the gallery

$$C_o = \{1\}, \{s_1\} = s_1 C_o, \{s_1 s_2\} = s_1 s_2 C_o \ldots, \{s_1 \ldots s_n\} = \{w\} = C_n$$

connects the two chambers. ♣

15.4 Configurations of three chambers

The following discussion is important in the sequel, and is of interest in its own right. It might be viewed as a significant exercise in understanding the geometry of a building, especially the contrast between *thickness* and *thinness*.

The first lemma asserts something possibly already clear, but worth repeating for clarity.

Let X be a thick building. Let C, C' be *(distinct) adjacent* chambers in X, and let D be a third chamber, distinct from C, C'. In this section, for two chambers x, y in X let $d(x, y)$ be the length of a minimal gallery from x to y. We will call this the *gallery distance* from x to y. The gallery distance $d(C', D)$ is either $d(C, D) + 1$, $d(C, D) - 1$, or $d(C, D)$, just because C, C' are adjacent.

Lemma: In a Coxeter complex $A = \Sigma(W, S)$, if C, C', D are chambers so that C, C' are distinct and s-adjacent, then $d(C', D) = d(C', D) \pm 1$. In particular, $d(C', D) \neq d(C, D)$.

Proof: Without loss of generality (since W acts transitively), we may take $C = \{1\}$, $C' = \{s\}$, and $D = \{w\}$. We know (3.4), (3.6) that minimal galleries from C to D are in bijection with reduced expressions for w. In particular, $d(C, D) = \ell(w)$. More generally, for any $w, w' \in W$, we have

$$d(\{w\}, \{w'\}) = \ell(w^{-1}w')$$

Then

$$d(C', D) = \ell(s^{-1}w) = \ell(sw) = \ell(w) \pm 1 = (C, D) \pm 1$$

This is the result. ♣

Proposition: If $d(C', D) = d(C, D) + 1$, then there is a minimal gallery γ from C' to D of the form

$$\gamma' = (C', C, \ldots, D)$$

In the opposite case where $d(C', D) = d(C, D) - 1$ there is a minimal gallery γ from C to D of the form

$$\gamma = (C, C', \ldots, D)$$

For $d(C', D) = d(C, D) \pm 1$, there is an apartment containing all three of the chambers. On the other hand, if $d(C, D) = d(C', D)$, then there is a chamber C_1 so that there are minimal galleries γ, γ' from C, C' to D of the form

$$\gamma = (C, C_1, \ldots, D)$$

$$\gamma' = (C', C_1, \ldots, D)$$

In this case there is *no* apartment containing all three chambers.

Proof: If $d(C;, D) = d(C, D) + 1$, then for any minimal gallery

$$\gamma = (C, C_1, \ldots, D)$$

from C to D, the gallery

$$\gamma' = (C', C, C_1, \ldots, D)$$

obtained by prefixing C' to γ is necessarily a minimal gallery from C' to D. And then by *convexity of apartments* (4.5), the minimal gallery from C' to D (which happens also to contain C) lies in any apartment containing C' and D. (There is at least one such apartment, by the building axioms (4.1)). The case $d(C', D) = d(C, D) - 1$ is symmetrical.

Now suppose that $d(C', D) = d(C, D)$. The previous lemma shows that the three chambers cannot lie in a common apartment. Let δ be the W-valued function defined above on pairs of chambers in X. Put $w = \delta(C, D)$ and $s = \delta(C, C')$. In particular, this means that C, C' are s-adjacent. Let

$$\gamma = (C = C_o, C_1, \ldots, C_n = D)$$

be a minimal gallery from C to D, of *type* $(s_1, \ldots, s_n)$. We saw just above that $w = s_1 \ldots s_n$ is a reduced expression for w since γ is minimal. Consider that gallery

$$\gamma' = (C', C, C_1, \ldots, D)$$

Since it is of length $n + 1$, which is longer by 1 than $d(C', D) = d(C, D)$, it is *not* minimal, so (from above) the word

$$(s, s_1, \ldots, s_n)$$

is *not* reduced. That is, $\ell(sw) < \ell(w)$. As a consequence of the Exchange Condition (1.7), we conclude that w has *some* reduced expression which begins with $s_1 = s$.

Since we have shown above (15.3) that there is a minimal gallery from C to D of type $(s_1, \ldots, s_n)$ for *every* reduced expression

$$s_1 \ldots s_n = w$$

for w, we conclude that there is a gallery

$$\gamma = (C = C_0, C_1, \ldots, C_n = D)$$

with $\delta(C, C_1) = s$. That is, C, C_1 are s-adjacent. But C' also shares the unique facet of C of type s, so the three chambers C, C', C_1 are mutually s-adjacent. In particular, with the gallery γ as just specified,

$$\gamma' = (C', C_1, \ldots, D)$$

is a minimal gallery from C' to D. The point is that γ, γ' differ only in that one begins with C while the other begins with C', as asserted in the proposition. ♣

Proposition: Let C, C', D be three distinct chambers, with C, C' being s-adjacent. Fix an apartment A containing C, C', and let ρ, ρ' be the *retractions* of X to A centered at C, C', respectively. Let H, H' be the half-apartments corresponding to the reflection s of A in which C, C' lie, respectively.

- If $d(C', D) > d(C, D)$, then $\rho D = \rho' D \in H$.
- If $d(C', D) < d(C, D)$, then $\rho D = \rho' D \in H'$.
- If $d(C', D) = d(C, D)$, then $\rho D \in H'$ and $\rho' D \in H$, and $s\rho D = \rho' D$.

Note that in the third of these possibilities, C and ρD are in opposite half-apartments, and C' and $\rho' D$ are in opposite half-apartments.

Proof: If $d(C', D) = d(C, D) + 1$, then by the previous proposition C, C', D lie in a common apartment B. Then B is mapped isomorphically to A by ρ, and ρ is the identity map on $A \cap B$: this was a fundamental property of these retractions (4.2). Then surely

$$d(C, \rho D) = d(C, D) < d(C', D) = d(C', \rho D)$$

Thus, by our corollary of Tits' theorem characterizing half-apartments by gallery distances (4.6), we conclude that ρD is in the half-apartment H of s in which C lies. Further, since B contains C', another fundamental property of the retractions ρ, ρ' is that $\rho|_B = \rho'_B$. Thus, we have the first assertion. The second assertion is symmetrical.

Now consider the case that $d(C, D) = d(C', D)$. Since ρ preserves gallery distances to C and cannot *increase* gallery distances to C' (4.2), we have

$$d(C, \rho D) = d(C, D) = d(C', D) \geq d(C', \rho D) \neq d(C, rhoD)$$

Thus, unavoidably $d(C', \rho D) < d(C, D)$, which implies that $\rho D \in H'$, again by the corollaries (4.6) to Tits' theorem. Symmetrically, $\rho' D \in H$.

Since these retractions are type preserving (4.4), we have

$$\delta(C, \rho D) = \delta(C, D)$$

and

$$\delta(C', \rho' D) = \delta(C', D)$$

where δ is the W-valued distance function used above in discussion of the existence of galleries of prescribed type (15.3). Now we invoke the previous proposition, to be sure that there is a gallery

$$(C_1, C_2, \ldots, C_n = D)$$

with C_1 adjacent to both C and C' and so that

$$\gamma = (C, C_1, C_2, \ldots, C_n = D)$$

and

$$\gamma' = (C', C_1, C_2, \ldots, C_n = D)$$

are both minimal galleries. From this the middle equality in the following is obtained:

$$\delta(C', \rho' D) = \delta(C', D) = \delta(C, D) = \delta(C, \rho D)$$

Thus, we deduce from the definition of δ that $\rho D = wC = \{w\}$. Similarly, letting $\rho' D = \{w'\}$, as $C' = \{s\}$, we have

$$w = \delta(C', \rho' D) = s^{-1} w' = sw'$$

so $w' = sw$. That is, $\rho' D = s\rho D$ as claimed. ♣

Remarks: The assertions of the previous propositions and lemma can be strengthened a little if the things learned about the W-valued function δ in the course of the proofs are included. However, we will not need these sharper statements in the sequel.

15.5 Subsets of apartments, strong isometries

The goal of this section is to give a sharp characterization of subsets Y of a thick building X so that Y lies inside some apartment in X (in the *maximal* apartment system). This will be done in terms of the notion of *strong isometry*, defined below in terms of the W-valued distance function δ used earlier (15.3) in discussion of existence of galleries of prescribed type. Let (W, S) be a Coxeter system so that the apartments in X are isomorphic to the Coxeter complex $\Sigma(W, S)$.

Let Y, Z be two sets of chambers in X. A **strong isometry** $\phi : Y \to Z$ is a bijection so that for all $C, D \in Y$ we have

$$\delta(\phi C, \phi D) = \delta(C, D)$$

Theorem: Let Y be a set of chambers in a thick building X. If Y is *strongly isometric* to a subset of some apartment, then Y *is* a subset of some apartment in the *maximal* apartment system for X.

Proof:

We need some auxiliary maps:

Proposition: For a chamber C in an apartment A, and for another chamber D, there is a unique label-preserving

$$\rho = \rho_{D;C,A} : X \to A$$

which sends D to C, and so that the restriction of ρ to any apartment B containing D is an isomorphism to A.

Proof: Uniqueness follows immediately from the Uniqueness Lemma (3.2).

For fixed apartment B containing D, we define ρ as follows. For an apartment B containing D, put

$$\rho = j_B \circ \rho_{B,D}$$

where $\rho_{B,D}$ is the canonical retraction of X to B centered at D, and where j_B is a label-preserving isomorphism $j : B \to A$ sending D to C. The retraction $\rho_{B,D}$ itself is an isomorphism when restricted to an apartment containing D, so ρ also has this property. ♣

Lemma: A *strong isometry* $f : Y \to A$ to an apartment A is determined by knowing fD for any single chamber $D \in Y$. In fact, f is nothing but the map $\rho = \rho_{D;C,A}$ of the previous proposition, restricted to Y.

Proof: Fixing an identification of A with a Coxeter complex $\Sigma(W, S)$ so that $fD = \{1\}$, the strong isometry property entails that if $\delta(D, D') = w$ then $\delta(fD, fD') = w$. But there is exactly one chamber $C' = \{w'\}$ in $\Sigma(W, S)$ so that

$$w = \delta(\{1\}, C')$$

namely $C' = \{w\}$, since $\delta(\{1\}, \{w'\}) = w'$.

Now we check that this map f agrees with $\rho = \rho_{D;C,A}$ restricted to Y. For this, we use the characterization of $\rho_{D;C,A}$ in the proposition just above. For another chamber D' in Y, let B be an apartment containing both D and D', and put $\delta(D, D') = w \in W$. Let γ be a gallery in B from D to D', of type $(s_1, \ldots, s_n)$. We have

$$s_1 \ldots s_n = \delta(D, D')$$

The map ρ is a label-preserving isomorphism, so the gallery $\rho(\gamma)$ in A from ρD to $\rho D'$ is of the same type, and we conclude that

$$\delta(\rho D, \rho D') = \delta(C, \rho D') = w$$

But, again, $C' = \{w\}$ is the only chamber in A so that $w = \delta(\{1\}, C') = w$. Thus, indeed, f and ρ agree on Y. ♣

The following lemma is the crucial point here.

Lemma: Let $f : Y \to A$ be a strong isometry to an apartment A in X. For any chamber C' *not* in the image $f(Y)$ of f but *adjacent* to a chamber in the image, there is a strong isometry

$$\tilde{g} : f(Y) \cup \{\bar{C}'\} \to X$$

extending the inverse

$$f^{-1} : f(Y) \to Y$$

of the map f.

Proof: We identify A with a Coxeter complex $\Sigma(W, S)$ in such manner that C corresponds to the chamber $\{1\}$.

Let C be the chamber in the image $f(Y)$ to which C' is adjacent, and suppose that these two chambers are s-adjacent. Let D be the chamber in Y which maps to C by f, and let D' be any chamber in X which is s-adjacent to D (and not equal to it). Let y be a chamber in Y and let $x = f(y)$.

Let B be an apartment containing both D and D'. Existence of this is assured by the building axioms (4.1). Let H, H' be the half-apartments for s in A containing C, C', respectively. Let j_B be the unique label-preserving isomorphism $B \to A$ sending D, D' to C, C', respectively. Let $J = j_B^{-1} H$ and $J' = j_B^{-1} H'$. These are half-apartments containing D, D', respectively.

Write

$$\rho = \rho_{D;A,C} = j_B \circ \rho_{B,D}$$

$$\rho' = \rho_{D';A,C'} = j_B \circ \rho_{B,D'}$$

From the considerations of the previous section (15.4), either $\rho_{B,D} y = \rho_{B,D'} y$ or $\rho_{B,D} y = s\rho_{B,D'} y$, with the latter possible only if $\rho_{B,D} y \in J'$ (and, concommitantly, $\rho_{B,D'} y \in J$).

The isomorphism j_B transports this to A. Thus, either $\rho y = \rho' y$, or else $\rho y = s\rho' y$, with the latter possible only if $\rho y \in H'$ and $\rho' y \in H$. That is, invoking the previous lemma, either $\rho' y = \rho y = x$ or possibly $\rho' y = s\rho y = sx$, and the latter is not possible unless $x \in H'$. Paraphrased, this is that either

$$(\rho' \circ f^{-1})(x) = x$$

or

$$(\rho' \circ f^{-1})(x) = sx$$

with the latter possible only if $x \in H'$.

If $f(Y) \subset H$, then we have seen that

$$f = \rho|_Y = \rho'|_Y$$

so we extend f by taking ρ' on $Y \cup D'$:

$$\rho' : Y \cup D' \to f(Y) \cup C'$$

where $C' \notin f(Y)$. Thus, we have the assertion of the lemma in this case.

On the other hand, if $f(Y)$ does *not* lie entirely inside H, then we *claim* that we can *choose* the chamber D' so that $\rho' y = fy$ (rather than $sf(y)$) for some $y \in Y$ so that $f(y) \in H'$. Indeed, if D' is initially chosen so that $\rho' y = sfy$, then (as above) it must be that there are minimal galleries γ, γ' from D, D' to y of the form

$$\gamma = (D, D_1, D_2, \ldots, D_n = y)$$

$$\gamma = (D', D_1, D_2, \ldots, D_n = y)$$

That is, they are the same except for beginning at D or D'. The chamber D_1 is adjacent to *both* D, D'. Replacing D' by D_1 in this scenario achieves the effect that $d(D', y) = d(D, y) - 1$, so (after this replacement) $\rho y = \rho' y \in H'$. Since (from the previous lemma) $f = \rho|_Y$, we have succeeded in arranging $fy = \rho' y \in H'$.

Now we claim that necessarily $fy_1 = \rho' y_1$ for *all* $y_1 \in Y$. Suppose, to the contrary, that there is $y_1 \in Y$ so that (instead) $sfy_1 = \rho' y_1$. Since all the maps are non-increasing on gallery lengths,

$$d(y, y_1) \geq d(\rho' y, \rho' y_1) = d(fy, sfy_1)$$

Let ϕ be the *folding* of A to itself which is a retraction to H', and maps H to H'. Let

$$\gamma = (fy = C_o, C_1, \ldots, C_n = sfy_1)$$

be a minimal gallery from $fy \in H'$ to $sfy_1 = \rho' y_1 \in H$. Since the gallery starts in H' and ends in H, there must be an index i so that $C_i \subset H'$ while $C_{i+1} \subset H$. Then $\phi C_{i+1} = C_i$, since these two chambers are adjacent across the wall corresponding to s. Then the gallery $\phi\gamma$ from fy to $fy_1 = s\rho' y_1$ *stutters*, so

$$d(fy, sfy_1) > d(fy, fy_1)$$

Putting this together, we have

$$d(y, y_1) > d(fy, fy_1)$$

This is impossible, since f is an isomorphism.

We conclude in this case as well that the strong isometry

$$f : Y \to f(Y) \subset A$$

can be extended to a strong isometry

$$\rho'|_{Y \cup D'} : Y \cup D' \to f(Y) \cup C' \subset A$$

This proves the lemma. ♣

Now we can prove the theorem. From the last lemma, if $f(Y) \subset A$ is not the whole collection Y_A of chambers in the apartment A, then we can extend f^{-1} to a strong isometry on $f(Y) \cup C'$ for some chamber C' adjacent to a chamber in $f(Y)$. Let Y_A be the set of all chambers in A, and let Φ be a *maximal* one among all strong isometries extending f^{-1} to maps from some set Y_o of chambers in A. If Y_o were a *proper* subset of Y_A, then the last lemma shows that we could further extend Φ, contradicting the maximality. Thus, this extension Φ must be a strong isometry defined on the whole collection Y_A of chambers in the apartment A.

Then $Y \subset \Phi(Y_A)$, and $\Phi(Y_A)$ is strongly isometric to the set of all chambers in an apartment via Φ^{-1}. Thus, we could have assumed at the outset that $f : Y \to Y_A$ was a strong isometry from Y to the set Y_A of *all* chambers in the apartment A.

Thus, from the discussion above of apartments in the *maximal* apartment system, *if* we can extend f to a label-preserving *chamber complex* map $\tilde{f} : \tilde{Y} \to A$ on the chamber complex $\tilde{Y}$ consisting of Y *and all faces* of chambers in Y, *then* we can conclude that $\tilde{Y}$ is an apartment in the maximal apartment system.

Fix a chamber $C \in Y$. If we identify A with a Coxeter complex $\Sigma(W, S)$, we may as well suppose that $f(C)$ is the chamber $\{1\} = \langle\emptyset\rangle$, and identify the facets of $f(C)$ with the generating set S of the Coxeter group W. Since f is a strong isometry, for each $w \in W$ there is exactly one $C_w \in Y$ so that $\delta(C, C_w) = w$, where δ is the W-valued 'distance' function on the whole building.

Then for a subset T of S and $w \in W$, we attempt to define f by

$$f(\bigcap_{w' \in w\langle T\rangle} C_w) = w\langle T\rangle)$$

For each $s \in S$ and chamber $f(C_w) = \{w\}$ in A, there is exactly one chamber in A which is s-adjacent to $f(C_w)$, namely $s\{w\} = \{ws\} = f(C_{ws})$. (Note that it is ws and not sw!) Therefore, since f respects δ, the chamber C_{ws} is the *unique* chamber in Y so that $\delta(C_w, C_{ws}) = s$. Computing δ by taking an apartment containing both C_w and C_{ws}, we see that they are *adjacent*, and thus that the intersection

$$F_{\{w,ws\}} = C_w \cap C_{ws}$$

is a facet (codimension-one face) of both.

Thus, we can at least extend f to *facets* by

$$f(F_{\{w,ws\}}) = \{w, ws\} = w\{1, s\} = w\langle s\rangle$$

Now *any* face of a chamber C_w can be expressed in a *unique* way as an intersection of facets of C_w, simply because all this takes place inside a simplicial complex. As just noted, these facets are all of the form $(F_{\{w,ws\}} = C_w \cap C_{ws}$ for $s \in S$. Then a face x of C_w has an expression of the form

$$x = \bigcap_{s\in T} F_{\{w,ws\}} = \bigcap_{s\in T} C_w \cap C_{ws} = C_w \cap \bigcap_{s\in T} C_{ws}$$

for a uniquely-determined subset T of S. That is, in particular, every face of C_w has a unique expression as an intersection of *chambers* in Y.

Thus, for a subset T of S, we can unambiguously define an extension by

$$f(\bigcap_{s\in T} C_{ws}) = \bigcap_{s\in T} \{w, ws\} = w\langle S\rangle$$

since the indicated intersection is in the *Coxeter complex*. This defines f on every face of every chamber from Y, by remarks above. And this extension preserves inclusions, as was verified for facets by the δ-preserving property, and then by construction for smaller faces. Thus, this extension is a simplicial-complex map of $\tilde{Y}$ to A.

Finally, every simplex in Y is certainly expressible in a (unique) manner as an intersection of facets of chambers in Y, so the extended version of f is a *surjection*. Thus, since the extension was already seen to be an *injection* on simplices in $\tilde{Y}$. Thus, the extension really is an *isomorphism* of simplicial complexes. This completes the proof of the theorem. ♣

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16. The Spherical Building at Infinity

- Sectors
- Bounded subsets of apartments
- Lemmas on isometries
- Subsets of apartments
- Configurations of chamber and sector
- Configurations of sector and three chambers
- Configurations of two sectors
- Geodesic rays
- The spherical building at infinity
- Induced maps at infinity

Affine buildings have natural spherical buildings associated to them by a sort of 'projectivization' process. The relationships between the two buildings have as consequences not only for the geometry of the affine building, but also for groups acting upon the buildings. This idea is the culmination of the study of affine buildings.

In the special case that the affine building is a *tree* (that is, is one-dimensional), the spherical building at infinity is called the set of **ends of the tree**.

16.1 Sectors

This section begins a slightly technical but essential further study of affine Coxeter complexes A, (or, more properly, of their geometric realizations $|A|$). This is most important in later construction of the *spherical building at infinity* attached to an affine building.

Let $A = \Sigma(W,S)$ be an affine Coxeter complex (3.4), (13.6), which we identify with its geometric realization $|A|$ (13.5). Let H be the collection of all hyperplanes fixed by reflections, so the hyperplanes in H are the *walls* in A (12.1), (12.4). We have shown (12.4) that there is a point x_o (which may as well be called 0) in A so that every hyperplane in H is parallel to a hyperplane in H passing through $x_o = 0$. Let $\bar{H}$ be the collection of hyperplanes in H through 0. We have shown (12.4), (13.2), (13.3), (13.6) that $\bar{H}$ is *finite*.

Further, we have shown (12.4),(13.6) that the hyperplanes in $\bar{H}$ cut A into *simplicial cones* c all with vertices at $x_o = 0$. For $x \in A$, a translate $x + c$ of one of these simplicial cones is called a **sector** in A with vertex x. The **direction** of the sector is c. If one sector $x' + c'$ is contained in another sector $x + c$, then $x' + c'$ is a called a **subsector** of $x + c$. Two sectors $x + c, y + d$ have **opposite direction** if $d = -c$.

The following lemma is essentially elementary, but we give the proof as another example of this genre of computation.

Lemma: The intersection

$$(x+c) \cap (y+c)$$

of two sectors with the *same direction* c is a sector $z+c$ with the same direction c. A subsector $x'+c'$ of $x+c$ has the same direction as $x+c$.

Proof: A simplicial cone c in an n-dimensional vectorspace is defined by n linear inequalities $\lambda_i > 0$ and that the λ_i are linearly independent. A sector $x+c$ is then defined by linear inequalities $\lambda_i > \lambda_i(x)$. Thus, the intersection of $x+c$ and $y+c$ consists of the set of points where

$$\lambda_i > \sup(\lambda_i(x), \lambda_i(y))$$

The fact that there are exactly n such inequalities and that the λ_i are linearly independent assures that there is a point z so that

$$\lambda_i(z) = \sup(\lambda_i(x), \lambda_i(y))$$

Then the intersection is just $z+c$, as desired.

Then each λ_i has a lower bound on a subsector $x'+c'$, so has a lower bound on c' itself, using linearity. But the only alternatives for the behavior of each λ_i on c' is that it be positive everywhere or negative everywhere, so every λ_i must be positive on c', and it must be that $c = c'$. ♣

Lemma: Let $x+c$ and $y-c$ be two sectors with opposite directions. Suppose that $x \in y-c$ (from which also follows $y \in x+c$). Let C, D be chambers so that $\bar{C}$ meets

$$y+c = (y-x) + (x+c)$$

and $\bar{D}$ meets

$$x-c = (x-y) + (y-c)$$

If E is a chamber so that $\bar{E}$ meets $(x+\bar{c}) \cap (y-\bar{c}$ then E occurs in some *minimal gallery* from C to D.

Proof: We show that no element of H separates E from *both* C and D. Let $\eta \in H$ be defined by a linear equation $\lambda = c_o$. By changing the sign of λ if necessary, we can suppose that $\lambda > 0$ on c.

If $\lambda > 0$ on E, then $\lambda(y) > 0$, as otherwise $\lambda < 0$ on $y - \bar{c}$, contradicting the fact that $\lambda > 0$ on E. Then $\lambda > 0$ on $y+c$, so $\lambda > 0$ on $C \subset y+c$. That is, η does *not* separate C from E if $\lambda > 0$ on E.

On the other hand, if $\lambda < 0$ on E, then we have the symmetrical and opposite argument. That is, if $\lambda < 0$ on E, then $\lambda(x) < 0$, or else $\lambda > 0$ on $x+\bar{c}$, contradicting the fact that $\lambda < 0$ on E. Then $\lambda < 0$ on $x-c$, so $\lambda < 0$ on $D \subset x-c$. That is, η does *not* separate D from E if $\lambda < 0$ on E.

Recall that we showed that, in a Coxeter complex every minimal gallery from one chamber to another crosses every *wall* separating them once and only once, and, further, a *non-minimal* gallery must cross *some* wall *twice*

(3.6). We have shown that if a wall separates E from either C or D, then it does *not* separate E from the other of the two. So if we take a minimal gallery

$$\gamma_1 = (C, C_1, \ldots, C_m, E)$$

from C to E and a minimal gallery

$$\gamma_2 = (E, D_1, \ldots, D_n, D)$$

from E to D, then the gallery

$$\gamma = (C, C_1, \ldots, C_m, E, D_1, \ldots, D_n, D)$$

obtained by splicing them together *does not cross any wall twice*. Thus, the gallery γ contains E and is *minimal*. ♣

16.2 Bounded subsets of apartments

The main point of this section is that the property of being a *bounded subset of an apartment* in an affine building does *not* depend upon the apartment system.

Let X be an affine building (14.1) and $|X|$ its geometric realization with the canonical metric $d(,)$ as constructed above (14.2). When we speak of a *bounded* subset Y of $|X|$, we mean that there is a bound for $d(x, y)$ as x, y range over Y.

For two chambers C, D in X, we define $\mathcal{H}(C, D)$ to be the union of the (geometric realizations of all faces of) all chambers lying in some minimal gallery from C to D. This is a combinatorial version of a **closed convex hull** of the two chambers C, D.

Theorem: A *bounded* subset Y of $|X|$ is contained in an apartment A in a given apartment system $\mathcal{A}$ if and only if there is a pair C, D of chambers in X so that $Y \subset \mathcal{H}(C, D)$.

Remarks: Recall that we proved earlier that every minimal gallery from a chamber C to another chamber D lies inside *every* apartment containing both C and D (4.5).

Proof: For notational simplicity, we may write X for the geometric realization.

Let Y be a bounded subset of an apartment A in an apartment system $\mathcal{A}$ in X. We certainly may enlarge Y by replacing it by the union of all (geometric realizations of) faces of simplices (in A) which it meets.

Take an arbitrary *direction* c in A, in the sense of the previous section. Then we claim that there are points x, y in A so that

$$Y \subset (x + c) \cap (y - c)$$

Indeed, for each linear inequality $\lambda_i > 0$ defining the simplicial cone c there are constants a_i, b_i so that on Y we have $a_i < \lambda_i < b_i$. Then take the point

x to be the point where, for all i, $\lambda_i(x) = a_i$. That there is *any* such point is due to the fact that (as noted in the previous section) the *directions* are really simplicial cones, defined by linearly independent linear inequalities. Likewise take y to satisfy $\lambda_i(y) = b_i$.

Then, applying the second lemma of the previous section (16.1), there are two chambers C, D in A so that every chamber E contained in Y lies inside $\mathcal{H}(C, D)$. Thus, Y lies inside $\mathcal{H}(C, D)$. This proves half of the desired result.

The other half of the assertion is true in general, without any assumption of affine-ness, and was proven earlier (4.5): every minimal gallery connecting two given chambers lies inside *every* apartment containing the two chambers. Thus, have characterized bounded subsets of apartments in a manner independent of the apartment system. ♣

16.3 Lemmas on isometries

This section contains some elementary results on isometries of Euclidean spaces and of subsets thereof. We give careful proofs of these results, even though they are essentially elementary exercises and eminently believable anyway.

Let E be n-dimensional Euclidean space with the *usual* inner product $(,)$, norm

$$|x| = (x, x)^{1/2}$$

and metric

$$d(x, y) = |x - y|$$

Recall that a collection $x_o, \ldots, x_N$ of $N + 1$ points in E is *affinely independent* if

$$\sum_i s_i x_i = \sum_j t_j x_j$$

implies that

$$(t_o, \ldots, t_N) = (s_o, \ldots, s_N)$$

for any $(N + 1)$-tuples so that $\sum_i t_i = 1$ and $\sum_i s_i = 1$. Equivalently, these points are affinely independent if and only if

$$\sum_i s_i x_i = 0$$

for $\sum s_i = 0$ implies that all s_i are 0.

Lemma: Let $x_o, \ldots, x_n$ be affinely independent points in E. For a given list $d_o, \ldots, d_n$ of non-negative real numbers, there is at most one point x in E so that $d(x, x_i) = d_i$ for all indices.

Proof: Write

$$x_i = (x_{i,1}, \ldots, x_{i,n})$$

If s, t were two points satisfying all these conditions, then for all $1 \leq i \leq n$ we have

$$2(t - x_o, x_i - x_o) = |t - x_i|^2 - |t - x_o|^2 - |x_i - x_o|^2 = d_i^2 - d_o^2 - |x_i - x_o|^2$$

Thus, by hypothesis, for $1 \leq i \leq n$

$$(t - x_o, x_i - x_o) = (s - x_o, x_i - x_o)$$

In particular, for $1 \leq i \leq n$

$$(s - t, x_i - x_o) = 0$$

By hypothesis the functions $t \to (t, x_i - x_o)$ for $1 \leq i \leq n$ are linearly independent linear functionals on E. Thus, $s - t = 0$. This proves that there is *at most* one such point. ♣

Lemma: Let $x_o, \ldots, x_N$ be points in E. Let M be the N-by-N matrix with $(i,j)^{\text{th}}$ entry

$$(x_i - x_o, x_j - x_o)$$

Then these points are affinely independent if and only if M is of rank N.

Proof: Let Ω be the n-by-N matrix with i^{th} column $x_i - x_o$. Then

$$M = \Omega^\top \Omega$$

So by elementary linear algebra the rank of M is the rank of Ω. So surely $N \leq n$ if the rank of M is N, etc. And the rank of Ω is N if and only if the $x_i - x_o$ (for $i \geq 1$) are linearly independent.

Suppose that the rank is N, so that the $x_i - x_o$ are linearly independent. If

$$\sum_i s_i x_i = \sum_j t_j x_j$$

with $\sum_i s_i = 1$ and $\sum_i t_i = 1$ then we subtract

$$x_o = \sum_i s_i x_o = \sum_i t_i x_o$$

from both sides and rearrange to obtain

$$\sum_{i \geq 1} s_i (x_i - x_o) = \sum_{j \geq 1} t_j (x_j - x_o)$$

The assumed linear independence yields $s_i = t_i$ for all $i \geq 1$. Since $\sum_i s_i = 1$ and $\sum_i t_i = 1$ it follows that also $s_o = t_o$. This proves the affine independence.

On the other hand, suppose that

$$\sum_{i \geq 1} c_i (x_i - x_o) = 0$$

were a non-trivial linear dependence relation. Let

$$c_o = -\sum_{i \geq 1} c_i$$

Then we have
$$\sum_i c_i x_i = 0$$
and now
$$\sum_{i \geq 0} c_i = 0$$
Thus, the x_i are not affinely independent. ♣

Lemma: Given affinely independent points $x_o, x_1, \ldots, x_n$ and given points $y_o, y_1, \ldots, y_n$ in Euclidean n-space E, if
$$d(x_i, x_j) = d(y_i, y_j)$$
for all pairs of indices i, j, then there is a *unique* isometry $\phi : E \to E$ which sends x_i to y_i for all indices i. Specifically, we claim that the isometry is the function ϕ defined by
$$\phi(\sum_i t_i x_i) = \sum_i t_i y_i$$
for all $(n+1)$-tuples $(t_o, \ldots, t_n)$ with $\sum t_i = 1$.

Proof: The relation
$$2(x_i - x_o, x_j - x_o) = |(x_i - x_o) - (x_j - x_o)|^2 - |x_i - x_o|^2 - |x_j - x_o|^2 =$$
$$= |x_i - x_j|^2 - |x_i - x_o|^2 - |x_j - x_o|^2$$
shows that the inner products of the vectors $x_i - x_o$ and $x_j - x_o$ is determined by the distances between the points. Let $M(x_o, \ldots, x_n)$ be the n-by-n matrix whose $(i, j)^{\text{th}}$ entry is
$$(x_i - x_o, x_j - x_o)$$
Then the previous remark implies that
$$M(\phi x_o, \phi x_1, \ldots, \phi x_n) = M(x_o, \ldots, x_n)$$
In particular, since the x_i are affinely independent the matrix $M(x_o, \ldots, x_n)$ is of rank n. And then it follows that the images ϕx_i are also affinely independent, since $M(\phi x_o, \ldots)$ is of full rank. (See lemma above).

Since the x_i are affinely independent, every point in the Euclidean space E has a unique expression as an affine combination of the x_i's, so the map ϕ is indeed defined on all of E, and is well-defined. We check that it preserves distances: to do so, we may as well take $x_o = y + o = 0$, since we could translate all these points to achieve this effect. Thus, from above, we know that
$$(x_i, x_j) = (y_i, y_j)$$
for all indices i, j.

We have
$$|\phi(\sum_i s_i x_i) - \phi(\sum_j t_j x_j)|^2 = |(\sum_i s_i y_i) - (\sum_j t_j y_j)|^2 =$$

$$= |\sum_i (s_i - t_i)y_i|^2 = \sum_{i,j}(s_i - t_i)(s_j - t_j)\,(y_i, y_j) =$$
$$= |\sum_i (s_i - t_i)x_i|^2 = \sum_{i,j}(s_i - t_i)(s_j - t_j)\,(x_i, x_j) =$$
$$= |\phi(\sum_i s_i x_i) - \phi(\sum_j t_j x_j)|^2 = |(\sum_i s_i x_i) - (\sum_j t_j x_j)|^2 =$$

by reversing the earlier steps. This verifies the distance-preserving property of ϕ.

The uniqueness follows immediately from the lemma above which noted that there is at most one point at prescribed distances from a maximal set of affinely independent points. ♣

Corollary: Any isometry of a Euclidean space E is an affine map.

Proof: Choose a maximal set $x_o, x_1, \ldots$ of affinely independent points in E, and invoke the previous lemma. The formula there makes it clear that the isometry is affine, to say the least. ♣

Corollary: Let X, Y be a subsets of a Euclidean space E. Let $\phi_o : X \to Y$ be an isometry. Then there is an isometry $\phi : E \to E$ extending ϕ_o. If X contains $n+1$ affinely independent points then there is a *unique extension*.

Proof: If X contains $n+1$ affinely independent points $x_o, \ldots, x_n$, then we are done, by defining ϕ as in the lemma just above. The uniqueness follows as above in this situation.

If X does not contain $n+1$ affinely independent points, then X lies inside an affine hyperplane ξ. From the lemmas above, it follows that Y also lies inside a hyperplane η. By translating if necessary, we may suppose that these hyperplanes are *linear*, that is, pass through 0. Translating further, we may suppose that $x_o = y_o = 0$. By induction on the dimension n, there is an isometry $\phi_1 : \xi \to \eta$ extending ϕ_o. and ϕ_1 is *linear*. Then take two unit vectors x_*, y_* in perpendicular to ξ, η, respectively, and extend ϕ_1 to the desired ϕ by defining

$$\phi(x_1 + tx_*) = \phi_1(x_1) + ty_*$$

where $x_1 \in \xi$ and where t is real. Since ϕ_1 is a linear isometry it is easy to check that ϕ is an isometry. ♣

16.4 Subsets of apartments

Recall that in discussing the finer *general* geometry of buildings, we showed that a subcomplex Y of a thick building is contained in an apartment in the maximal apartment system if and only if it is *strongly isometric* to a subset of an apartment, in a *combinatorial* sense (15.5). Now we will obtain a refined analogue of this for affine buildings, involving the canonical metric (14.2) on the geometric realization, and now using the notion of *isometry* in a more literal metric sense.

Unfortunately, this theorem is substantial not only when measured by its important, but also when measured by length of proof.

With some justification provided by the observation above (16.2) that the notion of *bounded subset of apartment* is independent of the apartment system in an affine building, we now suppose that the apartment system $\mathcal{A}$ is the *maximal* system of apartments in a thick affine building X. (Recall that we showed earlier that the union of two apartment systems is again an apartment system, so there is a *unique* maximal apartment system (4.4)).

Let $d(,)$ be the canonical metric (14.2) on the building. For this section, let E be a Euclidean space isometric to any and all the (geometric realizations of) apartments in X. Indeed, in the construction of the canonical metric we did show that all apartments are isometric to each other.

Theorem: Suppose that a subset Y of an affine building X is either *convex* or has *non-trivial interior* , and suppose that Y is isometric to a subset of the Euclidean space E. Then Y is contained in some apartment in the maximal apartment system in X.

Corollary: A subset of X is an apartment in the maximal system if and only if it is isometric to the Euclidean space E.

Proof: *(of corollary):* Suppose that a subset Y of X is isometric to E. Since isometries respect straight line segments, and since E certainly is convex, it follows that Y is convex. Then the theorem applies, so Y is contained in an apartment B. And B itself is isometric to E.

We claim that no proper subset E' of E is isometric to E. Indeed, in our detailed discussion of isometries of Euclidean spaces (16.3), we showed that for any two subsets Y, Z of E, any isometry $\phi : Y \to Z$ has an *extension* to an isometry $\tilde{\phi} : E \to E$. That is, $\tilde{\phi}|_Y = \phi$. If E' were a *proper* subset of E, then an isometry $\phi : E' \to E$ would have an extension $\tilde{\phi} : E \to E$ which would also be an isometry. But since $\phi E' = E$, this extension could not be injective, contradiction. This proves the claim, and the corollary. ♣

Proof: First, as in the general discussion of the finer geometry of buildings (15.5), for given chamber C in apartment A, and for another chamber D in the building, there is a unique chamber-complex map $\Phi : X \to A$ so that

$\Phi(D) = C$, and so that the restriction of Φ to any apartment containing D is an isomorphism to A. This Φ was constructed by composing the canonical retraction of X to any apartment B containing D with the isomorphism $B \to A$ taking D to C (and preserving labels). This map is essential in the proof.

Lemma: Suppose that the subset Y contains an open subset U of a chamber D, and that Y is isometric to a subset of the Euclidean space E. Let C be a chamber in an apartment A. Then there is a *unique* isometry $\phi : Y \to A$ so that

$$\phi|_U = \lambda_{D,C}|_U$$

where $\lambda_{D,C}$ is the geometric realization of the unique type-preserving simplicial complex isomorphism $D \to C$.

Proof: For uniqueness, let $\psi : Y \to A$ be an isometry, whose restriction to U is the same as the restriction of the type-preserving map $\lambda_{D,C}$. Then $\phi\psi^{-1}$ maps the subset $\psi(Y)$ of A to itself, and fixes $\psi(U)$ pointwise. The previous section (16.3) gives uniqueness, since U contains a maximal collection of affinely independent points.

For existence, let σ be an isometry $Y \to A$. Then $\sigma(U)$ and $\phi(U)$ are isometric subsets of A, and by the previous section (16.3) any isometry $\sigma(U) \to \phi(U)$ extends to an isometry τ of A to itself. The composite $\tau \circ \sigma$ is the desired isometry. ♣

The following is the essential extension trick in this whole argument. We abuse notation by writing X for the geometric realization of the thick affine building X.

Lemma: Suppose that Y contains the closure $\bar{D}$ of a chamber D in X. Suppose that ϕ is an isometry $\phi : Y \to A$ of Y to an apartment A, so that ϕ restricted to $\bar{D}$ is the (geometric realization of) the type-preserving simplicial complex isomorphism $\lambda_{D,C}$ of D to C. For any chamber C' in A adjacent to C, there is a chamber D' adjacent to D in X so that ϕ extends to an isometry

$$\tilde{\phi} : Y \cup \bar{D}' \to A$$

and so that the restriction of $\tilde{\phi}$ is the isomorphism $\lambda_{D',C'}$.

Remarks: In the preceding there is no assumption that C' is disjoint from the image of Y.

Proof: Let $\Phi : X \to A$ be the map mentioned at the beginning of the proof of the theorem, from (15.5), which takes D to C and gives an isomorphism $B \to A$ from any apartment B containing D. For a chamber D' adjacent to D, let Φ' be the analogous map $X \to A$ so that $\Phi'(D') = C'$ and so that Φ' is an isomorphism to A when restricted to any apartment containing D'.

From the previous lemma we know that ϕ is unavoidably the restriction of Φ to Y. What is necessary is to make a choice of the chamber D' adjacent to D so that *also* ϕ is the restriction of Φ' to Y. (These maps Φ, Φ' are type-preserving (4.4)).

Presuming that $C' \neq C$, let s be the label so that C' and C are $s-adjacent$. Let $D' \neq D$ be a chamber in X which is s-adjacent to D. Let η be the wall in A which separates C and C', with H the half-apartment in which C lies and H' the half-apartment in which C' lies.

In our discussion of the finer geometry of buildings in general, when looking at *configurations of three chambers* (15.4), we saw that for any chamber $y \in X$ either $\Phi' y = \Phi y$ or $\Phi' y = s\Phi y$. More precisely, letting $d_{\text{gal}}(x, y)$ the the gallery distance from one chamber x to another, there are three possibilities: If $d_{\text{gal}}(D', y) = d_{\text{gal}}(D, y)+1$, then $\Phi' y = \Phi y \in H$. If $d_{\text{gal}}(D', y) = d_{\text{gal}}(D, y)-1$, then $\Phi' y = \Phi y \in H'$. If $d_{\text{gal}}(D', y) = d_{\text{gal}}(D, y)$, then $\Phi' y = s\Phi y \in H$. Further, in the third case there are minimal galleries

$$\gamma = (D, D_1, D_2, \ldots, D_n = y)$$

$$\gamma' = (D', D_1, D_2, \ldots, D_n = y)$$

from D, D' to y, respectively. That is, in the third case there is a chamber D_1 in X which is s-adjacent to *both* D and D', and so that the minimal galleries agree except that one begins at D and the other at D'.

Thus, for all $y \in Y$ we do have $\Phi' y = \Phi y$ except possibly $\Phi' y = s\Phi y$, which can only happen if $\Phi y \in H'$, as in the previous paragraph. We claim that we can choose D' so that $\Phi' y = \Phi y$ for *all* $y \in Y$. Since Φ and ϕ agree on Y, this would prove the lemma.

If $\Phi Y \subset H$ then we are already done, since then Φ' must agree with Φ on Y, by the criteria just noted.

So suppose that the image $\phi Y = \Phi Y$ is not entirely contained in H. We need to check that in this case we can adjust D' so that *some* $z \in Y$ has the property that $\Phi' z = \Phi z$ even though $\Phi z \in H'$, the half-apartment containing C'. Indeed, if $\Phi' z = s\Phi z$ then $d_{\text{gal}}(D', y) = d_{\text{gal}}(D, y)$, then we replace D' by the chamber D_1. After this change,

$$d_{\text{gal}}(D', y) = d_{\text{gal}}(D, y) - 1$$

and (as recalled just above) we have $\Phi' z = \Phi z \in H'$.

Thus, we can suppose that we have $z_o \in Y$ so that $\Phi' z_o = \Phi z_o \in H'$, and can prove that $\Phi' z = \Phi z$ for *all* $z \in Y$. Suppose that $\Phi' z = s\Phi z \in H$ for some $z \in Y$. Note that Φ is an isometry on Y, and a fundamental property of the map Φ' is that it does not increase distances in the metric on X. (This was proven in the course of the construction (14.2) of the canonical metric on X). Let $[z_o, z]$ be the *straight line segment* in X from z_o to z. (In discussion of the metric on X we showed that the notion of straight line segment from one point to another makes sense and is intrinsic (13.7), (14.2)). Then

$$d(z_o, z) \geq d(\Phi' z_o, \Phi' z) = d(\Phi z_o, s\Phi z)$$

Suppose that we knew that for any two points $x, x' \in H'$ we had

$$d(x, sx') > d(x, x')$$

Then it would follow that

$$d(z_o, z) \geq d(\Phi z_o, s\Phi z) > d(\Phi z_o, \Phi z) = d(z_o, z)$$

contradiction.

Thus, to prove the lemma it suffices to prove that for any two points $x, x' \in H'$ we have $d(x, sx') > d(x, x')$. Happily, this is a very concrete question, unlike the relatively abstract combinatorial analogue we faced earlier in discussion of *general* buildings. That is, (geometric realization of the) the apartment A is a Euclidean space, the half-apartments H, H' are literal half-spaces, and the reflection s is a literal reflection.

To allay any doubts, we carry out this elementary computation: let e be a unit vector perpendicular to the wall (hyperplane) η, pointing in the direction of H'. Without loss of generality, we may suppose that $0 \in \eta$. Let $\langle , \rangle$ be the inner product on $A \approx E$. Then since $x, x' \in H'$ we have

$$\langle x, e\rangle > 0 \qquad \langle x', e\rangle > 0$$

The image sx' of x' is given by

$$sx' = x' - 2\langle x', e\rangle e$$

We compute the distance:

$$\begin{aligned} d(x, sx')^2 = |x - sx'|^2 = \langle x - sx', x - sx'\rangle &= \langle x, x\rangle - 2\langle x, sx'\rangle + \langle sx', sx'\rangle = \\ &= \langle x, x\rangle - 2\langle x, x'\rangle + 4\langle x, e\rangle\langle x', e\rangle + \langle x', x'\rangle = \\ &= |x - x'|^2 + 4\langle x, e\rangle\langle x', e\rangle > |x - x'|^2 \end{aligned}$$

where we use the fact that s preserves $\langle , \rangle$.

Thus, altogether, we have obtained the desired extension of the isometry. This proves the lemma. ♣

Now we prove a special case of the theorem, to which we will reduce the theorem afterward.

Lemma: If a subset Y of the building contains a closed chamber $\bar{C}$ and is isometric to a subset of the Euclidean space E. Then Y is contained in some apartment (in the maximal system).

Proof: In the general characterization of apartments in the maximal system (4.4), we showed that any simplicial subcomplex B which is isomorphic to an apartment by a label-preserving simplicial complex map is necessarily an apartment in the maximal system. We must obtain such a simplicial-complex isomorphism from the metric information we have here. And now we must exercise a little care to distinguish simplicial complex items from their geometric realizations.

Let A be an apartment containing C. From the lemma just above, there is an isometry $\phi : Y \to |A|$ fixing $\bar{C}$ pointwise. By the last lemma, we can repeatedly extend ϕ chamber by chamber as geometric realization $|\psi|$ of a (label-preserving) simplicial complex map ψ, in a manner consistent with the original map on Y. Thus, we obtain a label-preserving simplicial complex isomorphism ψ defined on some subcomplex Σ of X so that $Y \subset |\Sigma|$, $|\psi|$ restricted to Y is ϕ, and $\psi\Sigma = A$. By the result recalled in the previous paragraph, $\psi^{-1}A$ is an apartment in the maximal apartment system. ♣

Now we treat the general case of the theorem. By the last lemma, what needs to be shown is that the isometry $\phi : Y \to E$ can be extended to an isometry on a larger set containing a closed chamber.

In the case that Y has *non-empty interior* , necessarily Y contains an open subset of some chamber C lying inside an apartment A. We claim that (the geometric realization of) the canonical retraction $\rho_{A,C}$ of X to A centered at C gives an isometry of $Y \cup \bar{C}$ to A. Indeed, the first lemma above shows that $\rho_{A,C}$ maps Y isometrically to A. In the basic discussion (14.2) of the metric on an affine building we saw that such a retraction preserves distances from points in $\bar{C}$ (and of course is the identity on $\bar{C}$). This reduces this case of the theorem to the previous lemma, as desired.

Now consider the case that Y is *convex*. Let A be an apartment containing a chamber C so that a face x of C is *maximal* among simplices whose geometric realizations meet Y. Again we claim that the canonical retraction $\rho_{A,C}$ gives the desired isometry $Y \cup \bar{C} \to E$. In this case the issue is to see that $\rho_{A,C}$ preserves distances between points of Y. To this end, let $y \in x \cap Y$, and let p, q be two other points in Y, distinct from y.

Recall from the basic discussion (13.7), (14.2) of the metric that straight lines (geodesics) in $|X|$ are intrinsically defined, and are certainly preserved by isometries. Let p', q' be points on the straight line segments $[y,p], [y,q]$. By *convexity*, these geodesic line segments lie inside Y.

We claim that if p' is close enough to y then p' lies in x. Certainly p' close enough to y cannot lie in a proper face of x. Thus, if there were *no* neighborhood of y in $[y,p]$ which lay inside x, then points on $[y,p]$ near y would have to lie in a simplex $\tilde{x}$ having x as *proper* face, contradicting the maximality of x among simplices which meet Y. This proves the claim.

Thus, for p', q' on $[y,p], [y,q]$ near enough to y (but distinct from y) we have $p', q' \in x \subset \bar{C}$. Thus, $\rho = \rho_{A,C}$ does not move p', q' (in addition to not moving y).

Since an isometry takes straight lines to straight lines, and since on Y we have $\rho = \phi$, the points p', q' still lie on the straight lines $[y, \rho p], [y, \rho q]$, respectively. Further, the convex hull Δ of y, p, q must be mapped to the convex hull Δ' of $y, \rho p, \rho q$. Then the *angle* (inside $|A|$) between $[y, \rho p], [y, \rho q]$ must be the same as the angle between $[y, p'], [y, q]$, which is the original angle between $[y,p], [y,q]$.

Thus, by the *side-angle-side* criterion for congruence of triangles in Euclidean spaces (such as $|A|$), ρ must give an isometry of Δ to Δ'. In particular, the distance from ρp to ρq is the same as that from p to q.

From this, we conclude that ρ on $Y \cup \bar{C}$ is an isometry, allowing invocation of the previous lemma, and thus proving the theorem in this case as well. ♣

16.5 Configurations of chamber and sector

Here the possible relationships of an arbitrary chamber and an arbitrary sector inside a thick affine building are examined. The main point is the theorem just below. Still we look at the *maximal* apartment system $\mathcal{A}$ (4.4) in (the geometric realization of) an affine building X (14.1) with its canonical metric $d(,)$ (14.2). The existence theorem of this section is crucial in the ensuing developments.

A **sector** in X is a subset $\mathcal{C}$ of X which is contained in some apartment A and is a sector in A in the sense already defined (16.1).

Lemma: A sector $\mathcal{C}$ in X is a sector (in our earlier sense) in *any* apartment B in $\mathcal{A}$ which contains it.

Proof: Since $\mathcal{C}$ contains chambers, $A \cap B$ contains at least one chamber. Thus, from the axioms for a building (4.1), there is an isomorphism $\phi : B \to A$ fixing the intersection pointwise. Since $\mathcal{C}$ is a sector in A, $\phi^{-1}\mathcal{C} = \mathcal{C}$ is a sector in B. (And these maps have geometric realizations which are *isometries* (14.2)). ♣

Theorem: Given a chamber C in X and a sector $\mathcal{C}$ in X there is an apartment $B \in \mathcal{A}$ and a subsector $\mathcal{C}'$ of $\mathcal{C}$ so that both C and $\mathcal{C}'$ are contained in B.

Proof: Let A be any apartment containing $\mathcal{C}$. By the previous results on isometry criteria for sets Y to lie inside apartments (16.4), it would suffice to find a subsector $\mathcal{C}'$ of $\mathcal{C}$ and a chamber C' in A so that the canonical retraction $\rho = \rho_{A,C'}$ of X to A centered at C' (4.2) gives an isometry on $\mathcal{C}' \cup C$. Indeed, the inverse image of A under this isometry would be a subset of X isometric to an apartment, so would be an apartment itself, by the corollary to the theorem of the previous section (16.4).

From their construction (4.2), (14.2), these retractions do not *increase* distance: if $r > 0$ is large enough so that a ball (in X) centered at some point in A contains C, then $\rho_{A,C'}C \subset A$ is still contained in that ball, regardless of the choice of C'. Thus, there is a bounded subset Y of A in which the image of C by any retraction $\rho_{A,C'}$ lies.

Let $\mathcal{D}$ be a sector in A having direction *opposite* to the direction of $\mathcal{C}$ and containing Y. That there is such a sector is elementary, using only the (metric) boundedness of Y. Further, since the directions are opposite, we can arrange this $\mathcal{D}$ so that its base point x lies inside $\mathcal{C}$.

Take any chamber C' with $x \in \bar{C}'$, and take the sector $\mathcal{C}'$ in the direction of $\mathcal{C}$ but with base point x. We claim that this $\mathcal{C}'$ fulfills the requirements of the theorem. Let D be a chamber in A which meets $\mathcal{C}'$. Note in particular that this means that there is a point in the *open* simplex D which lies inside

$\mathcal{C}'$. It will suffice to show that $\rho = \rho_{A,C'}$ gives an isometry on $\bar{D} \cup C$ for any such D.

Since $\rho_{A,D}$ is an isometry on $\bar{D} \cup C$, it would suffice to show that $\rho_{A,D}|_C = \rho|_C$. To prove this, let γ be a minimal gallery from C' to C, and γ' a minimal gallery from C' to D. Let $\tilde{\gamma}$ be the gallery from D to C obtained by going from D to C' via γ' and then from C' to C via γ. Then $\rho\tilde{\gamma}$ is a gallery from D to ρC, which consists of going from D to C' via γ' (inside A) and then along $\rho\gamma$ from C' to ρC.

Since ρ preserves gallery distances from C' (4.2), $\rho\gamma$ is a minimal gallery from C' to ρC.

Earlier, in discussing sectors inside apartments (16.1), we proved a lemma which, as a special case, implies that *some* minimal gallery γ_m from D (which meets $\mathcal{C}'$) to ρC (which meets $\mathcal{D}$) includes C', since the closure $\bar{C}'$ of C' meets the intersection $\{x\} = \mathcal{C}' \cap \mathcal{D}$.

Certainly the part $\gamma_{m,1}$ of γ_m which goes from C' to ρC must be a minimal gallery from C' to ρC, and likewise the part $\gamma_{m,2}$ of γ_m which goes from D to C' must be minimal.

The point is that the gallery $\rho\tilde{\gamma}$ must also be minimal from D to ρC, since

$$\text{length } \rho\tilde{\gamma} = \text{ length } \gamma' + \text{ length } \rho\gamma =$$

$$= \text{ length } \gamma'_{m,1} + \text{ length } \rho\gamma_{m,2} = d_{\text{gal}}(D, C') + d_{\text{gal}}(C', \rho C)$$

Thus, since ρ cannot increase gallery distances, and preserves gallery distances from C', it must be that $\tilde{\gamma}$ is a minimal gallery from D to C.

Then, by the gallery-distance-preserving property of $\rho_{A,D}$, the image $\rho_{A,D}\tilde{\gamma}$ is *also* a minimal gallery from D to ρC.

So we have two mappings $\rho, \rho_{A,D}$ from $\tilde{\gamma}$ to the (thin chamber complex) A. Neither one sends $\tilde{\gamma}$ to a stuttering gallery, and they agree on γ'. Thus, by the Uniqueness Lemma (3.2), they must agree entirely. Thus, in particular, $\rho_{A,D}C = \rho C$, as desired. ♣

Corollary: Given a sector $\mathcal{C}$ in an affine building X, the union of all apartments containing a subsector of $\mathcal{C}$ is the whole building X. ♣

Corollary: Given a sector $\mathcal{C}$ in an apartment A in an affine building X, there is a unique chamber complex map $\rho_{A,\mathcal{C}} : X \to A$ so that on any apartment B containing a subsector $\mathcal{C}'$ of $\mathcal{C}$ the restriction $\rho_{A,\mathcal{C}}|_B$ is the isomorphism $B \to A$ (postulated by the building axioms).

Remarks: It is not clear (from either the statement of this corollary, or from its proof) what the relation of this retraction may be to the canonical retraction $\rho_{A,C}$ of X to A centered at a chamber C (4.2). But this does not concern us here.

Proof: Given an apartment B containing a subsector $\mathcal{C}'$ of $\mathcal{C}$, certainly $A \cap B$ contains a chamber. Thus, by the building axioms (4.1), there is an isomorphism $\phi_B : B \to A$ which gives the identity on $A \cap B$. We must check that for another apartment B' the maps ϕ_B and $\phi_{B'}$ agree on $B \cap B'$. Since both B, B' contain some subsector of $\mathcal{C}$, their intersection contains a subsector, so certainly contains a chamber. Let $\psi : B' \to B$ be the isomorphism (postulated by the axioms) which fixes $B \cap B'$.

Then $\phi_B \circ \psi$ is an isomorphism $B' \to A$, which agrees with $\phi_{B'}$ on a subsector of $\mathcal{C}$. By the Uniqueness Lemma (3.2), these two maps must be the same. This proves that $\rho_{A,\mathcal{C}}$ is well-defined.

The uniqueness assertion of the corollary follows from the Uniqueness Lemma (3.2). ♣

Corollary: Given a sector $\mathcal{C}$ in an apartment A, and given a chamber C in the affine building X, there is a subsector $\mathcal{C}'$ of $\mathcal{C}$ so that for any chamber C' meeting $\mathcal{C}'$ we have

$$\rho_{A,\mathcal{C}} C = \rho_{A,C'} C$$

Proof: Invoking the theorem, let $\mathcal{C}'$ be a small-enough subsector of $\mathcal{C}$ so that both $\mathcal{C}'$ and C are contained in a common apartment B. Then

$$\rho_{A,\mathcal{C}}|_B = \rho_{A,C'}|_B$$

by the Uniqueness Lemma, since these are isomorphisms which agree on the chamber C'. ♣

16.6 Configurations of sector and three chambers

This section develops some necessary properties of the retractions $\rho_{A,\mathcal{C}}$ attached to an apartment A and sector $\mathcal{C}$ within it, defined in the previous section (16.5).

Let X be a thick affine building (14.1). Let E be a Euclidean space to which all the (geometric realizations of) the apartments of X are isometric (13.6). Let A be an apartment containing a sector $\mathcal{C}$. Let ρ be the retraction $\rho_{A,\mathcal{C}}$ defined in the corollary to the theorem of the previous section. We recall that it is characterized by the property that on any apartment A' containing a subsector $\mathcal{C}'$ of $\mathcal{C}$ it gives an isomorphism to A which is the identity on $A \cap A'$.

Lemma: Let λ be an affine functional on an apartment A' in the thick affine building X which vanishes on a wall η in A'. Then either λ is *bounded above*, or is *bounded below* on the sector $\mathcal{C}'$. That is, either there is a constant λ_o so that $\lambda(z) \leq \lambda_o$ for all $z \in \mathcal{C}'$, or else there is a constant λ_o so that $\lambda(z) \geq \lambda_o$ for all $z \in \mathcal{C}'$.

Proof: (This is a reiteration of earlier ideas). Let Y be the collection of all hyperplanes in $A' \approx \Sigma(W,S)$ fixed by reflections in the Coxeter group W. Let $\bar{Y}$ be the collection of hyperplanes through a fixed point x_o in A' and parallel to some hyperplane in Y. Then, because (W,S) is *affine*, $\bar{Y}$ is *finite* (13.3), (13.4), (13.6). Let $\bar{\eta}$ be the hyperplane in $\bar{Y}$ parallel to the hyperplane

η on which λ vanishes. Then any one of the simplicial cones cut out by $\bar{Y}$ lies on one side or the other of $\bar{\eta}$, so λ is either positive or negative on every one.

Choose an isomorphism of A' to E, so that an origin is specified. Writing

$$\mathcal{C}' = x + c = (x - x_o) + (x_o + c)$$

where c is one of the simplicial cones cut out by $\bar{Y}$ and x is the vertex of $\mathcal{C}'$. Take $x' = x + h$ in $\mathcal{C}'$ with $h \in c$. If $\lambda > 0$ on c, then we have

$$\lambda(x') = \lambda(x + h) = \lambda(x) + \lambda(h) > \lambda(x)$$

If $\lambda < 0$ on c then we have

$$\lambda(x') = \lambda(x + h) = \lambda(x) + \lambda(h) < \lambda(x)$$

In either case we have the desired bound from one side. ♣

Corollary: Let η be a wall in an apartment A' containing a sector $\mathcal{C}'$. Then in *one* of the half-apartments cut out by η there is a bound for the maximum distance of any point of $\mathcal{C}'$ from η, while in the *other* half-apartment there is no such bound.

Proof: In the half-apartment where λ is bounded (whether from above or from below) the distance is bounded, while in the half-apartment where λ is unbounded the distance is bounded. ♣

Corollary: Given a sector $\mathcal{C}$ in an apartment A and given a wall η in A, there is a uniquely-determined half-apartment H cut out by η so that there is a subsector $\mathcal{C}'$ of $\mathcal{C}$ lying entirely inside H.

Proof: Let λ be an affine function vanishing on η. With given choice of origin in A, let the given sector be $x + c$ with x a point in A and c a simplicial cone. Change the sign of λ if necessary so that it is bounded *below* on $\mathcal{C}$. From the lemma, λ is necessarily positive on c. Let x_1 be any point in the half-apartment H where λ is positive. Then the subsector

$$x_1 + c = (x_1 - x) + (x + c)$$

of $x + c$ certainly lies inside H.

On the other hand, if $\lambda x_2 < 0$ for some point x_2 in A, then

$$x_2 + c = (x_2 - x_1) + (x_1 + c)$$

unavoidably meets H, since λ is unbounded positive on the sector $x_1 + c$. This proves the corollary. ♣

Thus, given *any* wall η in an apartment A containing $\mathcal{C}$, we can determine a notion of **positive half-apartment** cut out by η *determined by* $\mathcal{C}$ as being the half-apartment cut out by η containing some subsector of $\mathcal{C}$.

Proposition: Let $\mathcal{C}$ be a sector in an apartment A in the thick affine building X. Let $\mathcal{C}'$ be a subsector of $\mathcal{C}$ lying in the intersection $A \cap A'$ of A with another apartment A'. Let D_o, D, D' be three chambers with a common facet F, with D_o, D' lying in A'. Let $\rho : A' \to A$ be the retraction $\rho = \rho_{A,\mathcal{C}}$. Let η be the wall in A' separating D_o, D'. Suppose that D_o lies in the *positive half-apartment* determined by $\mathcal{C}'$ cut out by η. Then $\rho D = \rho D' \neq \rho D_o$.

Proof: Note that the proposition is not disturbed if we shrink the subsector $\mathcal{C}'$ further.

Let y_1 be a point in D_o. Let $\mathcal{C}_1$ be the sector in A' with the same direction as $\mathcal{C}$ (and $\mathcal{C}'$) with vertex at ρy_1. Shrinking $\mathcal{C}'$ if necessary, we can suppose that $\mathcal{C}'$ is a subsector of $\mathcal{C}_1$. By a corollary to the theorem of the previous section, we can shrink $\mathcal{C}'$ further so that for any chamber C in A meeting $\mathcal{C}'$ we have $\rho D_o = \rho_{A,C} D_o$.

Since we have arranged that $\mathcal{C}'$ lies entirely inside one half-apartment for η, the isomorphism $\rho : A' \to A$ sends $\mathcal{C}'$ to a subset of one half-apartment for $\rho\eta$. Since ρ is the identity map on $\mathcal{C}'$, it follows that $\mathcal{C}'$ is entirely within one half-apartment for $\rho\eta$ as well. This gives us a notion of *positive half-apartment* determined by $\mathcal{C}'$ for both η and $\rho\eta$. (The image $\rho\eta$ surely is itself a wall, because ρ is an isomorphism).

So the image ρy_1 under the isomorphism $\rho : A' \to A$ is in the positive half-apartment for the wall $\rho\eta$, since $\mathcal{C}' \subset \mathcal{C}_1$.

Let C be any chamber in A' which meets $\mathcal{C}'$. Note in particular that this means that there is a point in the *open* simplex C which lies inside $\mathcal{C}'$. Then C is necessarily also on the positive side of η. By the corollaries to Tits' theorem characterizing Coxeter complexes in terms of foldings ((3.6), the minimal gallery distance from C to D_o is less than the minimal gallery distance from C to D'. Thus, a minimal gallery

$$\gamma_o = (C = C_o, \ldots, C_{n-1} = D_o)$$

gives rise to a minimal gallery

$$\gamma' = (C = C_o, \ldots, C_{n-1} = D_o, D')$$

from C to D' by appending D' to γ_o.

From the general discussion of the finer combinatorial geometry of thick buildings, the *minimal* gallery γ' must be of *reduced type* (15.1). The gallery

$$\gamma = (C = C_o, \ldots, C_{n-1} = D_o, D)$$

obtained by replacing D' by D is of the same type as γ', since D_o, D, and D' have a common facet. Thus, the *reduced-type* gallery γ must be *minimal*.

Then the images $\rho_{A,C}\gamma$ and $\rho_{A,C}\gamma'$ are both necessarily minimal, since the retraction $\rho_{A,C}$ to A centered at C preserves gallery distances from C (4.2). In particular, $\rho\gamma$ and $\rho\gamma'$ are both *non-stuttering*, so $\rho D \neq \rho D_o$ and $\rho D' \neq \rho D_o$.

Since the retraction $\rho_{A,C}$ is also type-preserving (4.4), both ρD and $\rho D'$ have common facet (codimension one face) ρF with ρD_o. Since A is *thin*, we conclude that $\rho D = \rho D'$. ♣

16.7 Configurations of two sectors

Now the possible relationships two sectors inside a thick affine building are considered. The configuration studies of the previous sections are used here. The present study is the most delicate of all these.

Theorem: Given two sectors $\mathcal{C}, \mathcal{D}$ in a thick affine building X, there is an apartment $A_1 \in \mathcal{A}$ and there are subsectors $\mathcal{C}', \mathcal{D}'$ of $\mathcal{C}, \mathcal{D}$ respectively so that both $\mathcal{C}', \mathcal{D}'$ lie inside A_1.

Proof: (In the course of the proof we will review some aspects of affine Coxeter complexes which play a significant role).

Let E be a Euclidean space to which all the (geometric realizations of) the apartments of X are isometric (13.6). Let A, B be apartments containing $\mathcal{C}, \mathcal{D}$, respectively. We *identify* E with A. Let ρ be the retraction $\rho_{A,\mathcal{C}}$ attached to the sector $\mathcal{C}$ inside A (16.5). Again, it has the property that on any apartment A' containing a subsector $\mathcal{C}'$ of $\mathcal{C}$ it gives an isomorphism to A which is the identity on $A \cap A'$. Write $\mathcal{C} = x + c$ for some point $x \in A$ and a simplicial cone c.

The simplicial cone c is a chamber in the Coxeter complex $\Sigma(\bar{W}, \bar{S})$ attached to a finite Coxeter system $(\bar{W}, \bar{S})$. We recall how this comes about (13.2), (13.3), (13.6). Fixing a choice of *origin* 0 in $E = A$, let $w \to \bar{w}$ be the map which takes an affine transformation $w \in W$ of E to its *linear part* $\bar{w}$ with respect to the choice of origin. Then $\bar{W}$ is the image of W under $w \to \bar{w}$, and is a *finite* (Coxeter) group. For every hyperplane η fixed by one of the reflections in W, let $\bar{\eta}$ be a hyperplane in E parallel to η but passing through 0. Then the collection $\bar{S}$ of reflections through the hyperplanes $\bar{\eta}$ is a set of generators for $\bar{W}$, and $(\bar{W}, \bar{S})$ is a *finite* Coxeter system. Let

$$\bar{A} = \Sigma(\bar{W}, \bar{S})$$

(We showed (13.6) that an indecomposable Coxeter system, with Coxeter matrix positive semi-definite but not definite, gives rise to a *locally finite affine reflection group*, which is the sort of Coxeter group W we are considering at present. Indeed, this was the definition of *affine Coxeter complex*. The Perron-Frobenius lemma (13.3) was what proved that $\bar{W}$ is finite.)

Given a chamber D in B, let β_D be the unique label-preserving *isomorphism* $B \to A$ which takes D to ρD (15.5). Then $\beta_D \mathcal{D}$ is a sector in A, which by definition can be written as $x' + c'$ for some vertex x' and some chamber c' in the finite Coxeter complex $\bar{A}$ (which here appears as simplicial cones with vertex at x).

We say that c' is the *direction of $\mathcal{D}$ at D*, and write

$$c(\mathcal{D}, D) = c'$$

for this function.

Let $\bar{d}(c, c')$ be the minimal-gallery-length distance between two chambers c, c' in the *finite* Coxeter complex $\bar{A}$. Since $\bar{W}$ is *finite*, the gallery length

$$\bar{d}(c, c(\mathcal{D}, D))$$

achieves a maximum as D varies over chambers in B which meet $\mathcal{D}$. Let D_o be a chamber meeting $\mathcal{D}$ which realizes the maximum, and fix a point y_o inside D_o. Let $\mathcal{D}'$ be the subsector of $\mathcal{D}$ with vertex y_o.

By a corollary to the theorem of the section on configurations of chamber and sector (16.5), there is a subsector $\mathcal{C}'$ of $\mathcal{C}$ so that for any chamber C in A meeting $\mathcal{C}'$, we have $\rho D_o = \rho_{A,C} D_o$. Shrinking $\mathcal{C}'$ further if necessary, we can suppose that $\mathcal{C}'$ is a subsector of $\rho y_o + c$.

By results on metric characterization of apartments (16.4), it suffices for us to show that ρ is an isometry on $\mathcal{C}' \cup \mathcal{D}'$. That ρ restricted to $\mathcal{C}' \subset \mathcal{C}$ is an isometry is immediate. What needs to be compared are pairs of points in $\mathcal{D}$ and also pairs of points with one in $\mathcal{C}'$ and one in $\mathcal{D}'$.

Let D be a chamber in B meeting $\mathcal{D}'$, and take $y \in D \cap \mathcal{D}'$. In particular, this means that y is in the *interior* of the simplex D. Consider the straight line $[y_o, y]$. As in our discussion of reflection groups (12.1), (in effect invoking simply the local finiteness of the set of reflecting hyperplanes (13.2), (13.4)), it is possible to adjust y slightly so that the geodesic line $[y_o, y]$ does not intersect any faces of codimension greater than 1. Then we can unambiguously determine a sequence $D_o, D_1, \ldots, D_n = D$ of chambers in B so that $[y_o, y]$ passes through (the geometric realizations of) these chambers, and does so in the indicated order. And the adjustment assures that $\gamma = (D_o, \ldots, D_n)$ is a *gallery* from D_o to D.

Since a line cannot meet a hyperplane in more than one point (unless it is contained entirely within it), $[y_o, y]$ meets no wall twice. Thus, the gallery crosses no wall twice. Thus, this gallery is a *minimal* one from D_o to D. (Recall that a minimal gallery from one chamber to another *must* cross all the walls separating the two chambers, but need cross no more (3.6). This is true in general, without the assumption that we are in an *affine* building).

Next we claim that $\rho\gamma$ is *non-stuttering*, and that for any chamber C in $\mathcal{C}'$ we have $\rho D_i = \rho_{A,C} D_i$. We prove this by induction on the length n of the gallery.

By induction, suppose the assertion of the claim is true for

$$\gamma' = (D_o, \ldots, D_{n-1})$$

Then ρ is an isometry on

$$\xi = \mathcal{C}' \cup D_o \cup \ldots \cup D_{n-1}$$

By the metric characterization of apartments and their subsets (16.4), since ρ maps to the apartment A, ξ is contained in some apartment A'. Since A' contains a subsector of $\mathcal{C}$, by its construction ρ gives an isometry of A' to A. Further, since $A' \cap A$ contains any chamber C inside $\mathcal{C}'$, a fundamental characterization of the retraction $\rho_{A,C}$ is that it gives an *isomorphism* of A' to A (4.2).

If D_n already lies in A', then we have completed the induction step. So suppose that D_n does *not* lie in A'.

Let F be the common facet of $D_{n-1} \cap D_n$. Since D_n is *not* in A', there is a chamber D'_n in A', distinct from both D_{n-1}, D_n, and adjacent to D_{n-1} along F. Let η be the wall in A' separating D_{n-1} and D'_n.

Consider the case that D_o is in the positive half-apartment determined by $\mathcal{C}'$ for η in A' (16.6). From the corollaries to Tits' theorem characterizing Coxeter complexes in terms of walls and foldings (3.6), it must be that D_{n-1} is also on the positive side of η, since the gallery distance from D_o to D_{n-1} is one less than the gallery distance from D_o to D'_n.

Then we apply the proposition of the previous section (16.6) to the trio of chambers D_{n-1}, D_n, D'_n, with the notation otherwise identical. We conclude that $\rho D_n = \rho D'_n \neq \rho D_{n-1}$. Since $\rho_{A,C} D_{n-1} = \rho D_{n-1}$, this verifies the claim in case D_o is on the positive side of η.

Now we show that the choice of D_o guarantees that D_o *is* in the positive half-apartment for η in A'.

Suppose that D_o is on the *negative* side of η, as determined by $\mathcal{C}'$. As in the previous case, it follows that D_{n-1} is also on the negative side of η, while D'_n is on the positive side. In this case, the proposition of the previous section (16.6) can be applied again to the trio D_{n-1}, D_n, D'_n, but now with the roles of D_{n-1} and D'_n reversed from the previous case. Then we can conclude that $\rho D_{n-1} = \rho D_n$. We will reach a contradiction from this based on our choice of D_o, thereby completing the induction step.

Assume that $\rho D_{n-1} = \rho D_n$ as in the previous paragraph. For $i > 0$, let $y_i \in D_{i-1} \cap D_i$ be the point where $[y_o, y]$ crosses the hyperplane separating these two chambers. (Recall that we had adjusted y slightly so as to assure that there is just one such point, etc.). We had

$$\rho[y_o, y_1] \subset \rho D_o \subset \rho y_o + c'$$

for a simplicial cone c' (a $\bar{W}$-chamber).

By induction hypothesis, ρ is an isometry on the closure of $D_o \cup \ldots \cup D_{n-1}$, so ρ maps the subsegment $[y_o, y_n]$ to a straight line.

At the same time, we saw just above that $\rho D_n = \rho D_{n-1}$, so the straight line segment $[y_{n-1}, y]$ crossing from D_{n-1} to D_n is *not* mapped to a straight line segment under ρ. Indeed, ρy_n lies on the part of $\rho\eta$ touching the boundary of $\rho D_{n-1} = \rho D_n$, while ρy_{n-1} is on some *other* face, and ρy is in the *interior* . Yet ρ *does* give an isometry on the closure of each chamber, so the line segments $[y_i, y_{i+1}], [y_n, y]$ *are* mapped to straight line segments again.

Let s be the reflection in A across the hyperplane $\rho\eta$. We want to verify that $[\rho y_{n-1}, \rho y_n] \cup [\rho y_n, s\rho y]$ really does form the straight line $[\rho y_{n-1}, s\rho y]$. To see this, we let β be the unique label-preserving isomorphism from the apartment B containing $D_{n-1} \cup D_n$ to the apartment A' containing $\rho D_{n-1} = \rho D_n$ which sends D_{n-1} to ρD_{n-1}. Then β must map D_n to the *other* chamber in A' adjacent to ρD_{n-1} along $\rho(D_{n-1} \cap D_n)$. We have seen, in discussing the metric on affine buildings (14.2), that such an isomorphism must give an *isometry*. Thus, β preserves straight lines:

$$\beta[y_{n-1}, y] = [\beta y_{n-1}, \beta y]$$

Since $\beta D_n = s\rho D_n$, it must be that $s\rho[y_n, y] = \beta[y_n, y]$. Since

$$\beta[y_{n-1}, y_n] \cup \beta[y_n, y] = [\beta y_{n-1}, \beta y] = [\rho y_{n-1}, s\rho y]$$

it must be that ρy_n does really lie on the straight line between ρy_{n-1} and $s\rho y$.

Thus, the line segment $[\rho y_n, s\rho y]$ is a subsegment of $[\rho y_o, s\rho y]$. In effect, we had defined the simplicial cone (or $\bar{W}$-chamber) c' so that $\rho y_o + c'$ contains the segment $[\rho y_o, \rho y_1]$. Thus, $\rho y_n + c'$ contains $[\rho y_n, s\rho y]$.

Since ρy_n is *on* the hyperplane $\rho\eta$, and since ρy is on the *negative* side of $\rho\eta$, necessarily $s\rho y$ is on the *positive* side of η.

Let $\bar{s}$ be the *linear part* of s, that is, the image of s in the quotient group $\bar{W}$ of W. Then the *direction* $c(\mathcal{D}, D_n)$ of $\mathcal{D}$ at D_n is (from the definition above) $\bar{s}c'$, where c' is the direction of $\mathcal{D}$ at $D = D_o$ as above.

We had assumed that the gallery distance from c to c' was *maximal* obtainable as $c' = c(\mathcal{D}, D)$ in the spherical (that is, finite) Coxeter complex $\bar{A} = \Sigma(\bar{W}, \bar{S})$. Yet the assumption that c and c' are both on the same side of the wall defined by $\bar{s}$ implies that the gallery distance from c to $\bar{s}c'$ is strictly greater than the gallery distance from c to c', by corollaries to Tits' theorem characterizing Coxeter complexes by walls and foldings (3.6).

Hence, we have arrived at a contradiction to the assumption that D_o was on the negative side of the wall η. That is, we have shown that only the first case here, wherein D_o is on the *positive* side, can occur. *Thus, the induction step is completed, and the claim is proven.*

Now we can finish the proof of the theorem. Let β be the unique label-preserving isomorphism $\beta : B \to A$ and taking D_o to ρD_o. Since $\rho|'_{\mathcal{D}} = \beta|'_{\mathcal{D}}$, the Uniqueness Lemma (3.2) shows that $\beta = \rho$ on all of $\mathcal{D}'$. Thus, on $\mathcal{D}'$, ρ is an isometry.

Further, since (by the claim) for C in $\mathcal{C}'$ the map ρ coincides with $\rho_{A,C}$, which itself preserves distances from C, we see that ρ preserves distances between points of $\mathcal{C}'$ and points of $\mathcal{D}'$. This proves the theorem. ♣

16.8 Geodesic rays

This section brings into play all the previous results on affine buildings, including both combinatorial and metric structure. Throughout, the thick affine building X is assumed equipped with the *maximal* (that is, *complete*) system of apartments (4.4). Also, as has been done above, the distinction between a simplicial complex and its geometric realization is suppressed.

A **ray** $\mathbf{r}$ in the geometric realization X of a thick affine building X is a subset of X isometric to the half-line $[0, \infty)$. Let $\phi : [0, \infty) \to X$ be such an isometry. The image $\phi(0)$ is the **basepoint** or **vertex** or **origin** of $\mathbf{r}$, and the ray **emanates** from $\phi(0)$.

Since a ray $\mathbf{r}$ is *convex*, we know from the metric characterization of subsets of apartments (16.4) that a ray is contained in at least one apartment A. Since A is a Euclidean space, we conclude that the ray must be a ray in A in the most prosaic sense. That is, there is $x_o \in A$ and a vector v so that

$$r = \{x_o + tv : t \geq 0\}$$

More intrinsically, if we wish to invoke only the *affine* structure on A rather than using a choice of origin, we can write the ray as a set of affine combinations

$$\mathbf{r} = \{(1-t)x + ty : t \geq 0\}$$

for some x, y.

Before getting to the main point of this section, we look more carefully at the elementary aspects of the geometry of geodesic line segments.

Lemma: Let x, y, z be distinct points in X. Then for y' on $[x, y]$ close enough to x (but $y' \neq x$) and for z' close enough to x on $[x, z]$ (but $z' \neq x$) there is an apartment A so that both line segments $[x, y'], [x, z']$ lie inside A. Indeed, either both $[x, y']$ and $[x, z']$ lie inside the closure of a single chamber, or there are two adjacent chambers C, D the union of whose closure contains both $[x, y'], [x, z']$.

Proof: FIrst, we claim that for y' close enough to x on $[x, y]$ there is a chamber C whose closure $\bar{C}$ contains $[x, y']$. Let σ be the (open) simplex in which x lies. Then (by continuity) for y' sufficiently near x on $[x, y]$ it cannot be that y' lies in a proper face of σ. Thus, y' sufficiently near x lies in a simplex τ of which σ is a (possibly improper) face. Then the closure of τ contains x and is convex, so contains $[x, y']$. This proves the claim.

Let C, D be chambers whose closures contain some segments $[x, y'], [x, z']$, respectively. By the building axioms (4.1), there is an apartment A containing both these chambers, so containing their closure, so containing both these line segments. ♣

Proposition: Let $\mathbf{r}, \mathbf{s}$ be two rays emanating from a common point x. Then there is an angle θ so that for any apartment A containing line segments $[x, y], [x, z]$ of non-zero length inside $\mathbf{r}, \mathbf{s}$, respectively, the angle between $[x, y]$ and $[x, z]$ is θ. Further, let y_s, z_t be the points on $\mathbf{r}, \mathbf{s}$ at distance $s, t(respectively)$ from x. A *cosine inequality* holds:

$$d^2(y_s, z_t) \geq s^2 + t^2 - 2st \cos \theta$$

For each pair of values s, t, strict inequality holds unless x, y_s, z_t all lie in a common apartment.

Proof: By the previous lemma, there is at least *one* apartment A which contains some line segments $[x, y], [x, z]$ as indicated. Suppose another apartment B contains some segments $[x, y'], [x, z']$ on both rays. By shrinking the segments, we suppose that $[x, y], [x, z]$ lie inside both apartments A, B.

Then also the straight line $[y, z]$ lies inside both apartments, since quite generally $[y, z]$ lies inside any apartment containing both y, z. For that matter, for any pair of points p, q on any of the three segments $[x, y], [x, z], [y, z]$, the segment $[p, q]$ lies insides both A and B. Thus, the convex hull Δ of x, y, z lies inside both A, B.

We compute the angle θ at the vertex x by elementary Euclidean geometry: letting $\langle , \rangle$ be the usual inner product,

$$\cos\theta = \frac{\langle y - x, z - x\rangle}{|y - x| \cdot |z - x|} =$$

$$= \frac{|y - x|^2 + |z - x|^2 - |y - z|^2}{2 \cdot |y - x| \cdot |z - x|}$$

In particular, we see that once we have the triangle with vertices x, y, z inside a Euclidean space then the angles are determined by the edge lengths. This proves our claim that the angle is well-defined.

Thus, if the three points x, y_s, z_t *do* lie in a common apartment, we have the desired equality. What we must show is that the inequality holds more generally, and that the equality only occurs for all three points in an apartment.

Let C be any chamber in A whose closure contains x, and let ρ be the retraction of X to A centered at C (4.2). Recall that a fundamental metric property of ρ is that it preserves distances to x, and cannot increase distances between *any* two points y_s, z_t (14.2).

Thus, we can rearrange the inner product formula for the cosine of the angle to obtain

$$d(y_s, z_t) \geq d(\rho y_s, \rho z_t) = |\rho y - x|^2 + |\rho z - x|^2 - 2|\rho y - x||\rho z - x| \cos\theta =$$

$$= |y - x|^2 + |z - x|^2 - 2|y - x||z - x| \cos\theta$$

On the other hand, if the equality *does* hold then ρ gives an isometry on $\bar{C} \cup \{y_s, z_t\}$. We proved a theorem asserting that subsets of X which are either *convex* or *contain an open subset of an apartment* and which are isometric to a subset of Euclidean space lie inside an apartment. While the set $\{x, y_s, z_t\}$ did not meet this hypothesis, the larger set $\bar{C} \cup \{y, z\}$ *does*. This finishes the proof. ♣

Let $d(,)$ be the canonical metric on X. Two rays $\mathbf{r}, \mathbf{s}$ are **parallel** if there is a bound b so that, for every $x \in \mathbf{r}$ there is $y \in \mathbf{s}$ so that $d(x, y) \leq b$, and for every $y \in \mathbf{s}$ there is $x \in \mathbf{r}$ so that $d(x, y) \leq b$. This is visibly an equivalence relation.

If two rays $\mathbf{r}$ and $\mathbf{s}$ lie in a common apartment A, then elementary Euclidean geometry shows that they are parallel if and only if there is a translation in

A carrying one to the other. It is not so easy to see what happens inside the *building*, but we have the following (*provable*) analogue of a *parallel postulate*:

Proposition: Given $x \in X$ and given a ray $\mathbf{r}$ in X, there is a unique ray $\mathbf{s}$ *emanating* from x and *parallel* to $\mathbf{r}$.

Proof: Let A be an apartment containing $\mathbf{r}$. Let $\mathcal{C}$ be a sector in A with vertex the same as the vertex of $\mathbf{r}$. From the discussion of configurations of sectors and chambers (16.5) we know that there is a subsector $\mathcal{C}'$ of $\mathcal{C}$ so that both $\mathcal{C}'$ and x lie in some apartment A'.

Since $\mathcal{C}'$ is a translate within A of $\mathcal{C}$, its closure contains a ray $\mathbf{r}'$ parallel to $\mathbf{r}$. Then within A' we can translate $\mathbf{r}'$ so that its basepoint is at x, as desired. This proves *existence* of the ray parallel to $\mathbf{r}$ emanating from x.

To prove uniqueness of this ray, suppose that $\mathbf{r}, \mathbf{s}$ are distinct parallel rays with the same origin x. Since $\mathbf{r} \cap \mathbf{s}$ is non-empty (containing x) and closed and convex, it is a straight line segment $[x, y]$ for some point y. (Recall that from the discussion of the canonical metric on X (14.2) it follows that this straight line segment is intrinsically defined). If we replace $\mathbf{r}, \mathbf{s}$ by their subrays starting just at y, then we can suppose that $\mathbf{r} \cap \mathbf{s}$ is just their common basepoint y.

Now we invoke the *cosine inequality*

$$d^2(z_s, w_t) \geq s^2 + t^2 - 2st \cos \theta$$

proven just above, for the points z_s, w_t distances s, t out on the rays $\mathbf{r}, \mathbf{s}$, respectively. For fixed $s > 0$, as t varies, if $\theta \geq \pi/2$ then the minimum value of the right-hand side is s^2 achieved when $t = 0$. If $\theta < \pi/2$, then the minimum is $s^2 \sin^2 \theta$, achieved when $t = s \cos \theta$. Either way, we see that there is *no* absolute bound upon $d(z_s, \mathbf{s})$ as $s \to \infty$. This contradicts the assumption of parallelism. This proves uniqueness. ♣

16.9 The spherical building at infinity

Now everything is prepared for construction of the spherical building at infinity attached to a thick affine building X. As usual, we will also write X for the geometric realization of X. All references to apartments are with respect to the *maximal* apartment system.

A **point at infinity** or **ideal point** of X (or, most properly, an ideal point of X) is equivalence class of rays, under the equivalence relation of parallelism (16.8). Let X_∞ be the set of ideal points of X. By the proposition of the last section (16.8), for each point ξ at infinity, and for each $x \in X$, there is a unique geodesic ray with vertex x and in the parallelism class ξ. We will denote this geodesic ray by

$$[x, \xi)$$

and sometimes say that $[x, \xi)$ has **direction** ξ, or similar things.

Let A be an apartment. We know (4.3) that A is isomorphic as chamber complex to a Coxeter complex $\Sigma(W, S)$, and that the isomorphism class of the latter does not depend on *which* apartment (4.4). Further, the geometric realization of $\Sigma(W, S)$ is a Euclidean space E (13.6).

Let Y be the set of walls in $\Sigma(W, S)$, with respect to W. That is, Y is the set of hyperplanes fixed by a (generalized) reflection in W (1.6), (12.4), (13.6). Fix a point x in the geometric realization E, and let Y_x be the collection of all hyperplanes through x which are parallel to some hyperplane in Y. From the basic discussion of affine Coxeter systems, Y_x is *finite*, that is, there are only finitely-many parallelism classes of hyperplanes in Y (13.3), (13.4), (13.6). For each $\eta \in Y_x$, let λ_η be a non-zero affine functional on E which vanishes on η.

As in our discussion of (finite) reflection groups (12.1), (13.2), the set

$$\mathcal{C} = \{y \in E : \lambda_\eta y > 0 \quad \forall\, \eta\}$$

is the *fundamental conical cell*. (We also call it a *sector* as above). As seen earlier, the hypothesis that X is affine requires implicity that (W, S) is *indecomposable*, and that this implies $\mathcal{C}$ is a *simplicial cone* (13.6).

Just as we did with geodesic rays, we first give a definition of **conical cell** which does not depend on reference to apartments, but then observe that necessarily all conical cells lie inside apartments (in the *maximal* apartment system). The latter fact makes serious use of results above giving metric characterization of subsets of apartments in the maximal system (16.4).

Generally, for a partition $P = (Y_+, Y_-, Y_o)$ of Y_x into three (disjoint) pieces

$$Y_x = Y_+ \sqcup Y_o \sqcup Y_-$$

define a **conical cell** $c = c_P$ inside the Euclidean space $E = |\Sigma(W, S)|$ as the set of $z \in A$ such that

$$\lambda_\eta z > 0 \quad \text{for} \quad \eta \in Y_+$$

$$\lambda_\eta z < 0 \quad \text{for} \quad \eta \in Y_-$$

$$\lambda_\eta z = 0 \quad \text{for} \quad \eta \in Y_o$$

A **conical cell** in the building X is a subset of X isometric to a conical cell in E. Since the conical cells in E are convex, the metric characterization of subsets of apartments (16.4) implies that a conical cell c in X lies inside some apartment A. Then inside A the conical cell can be described by analogous inequalities specified by a partition (Y_+, Y_-, Y_o) of Y_x, as just above, but now of course with reference to affine functionals on A.

Another conical cell d in an apartment A corresponding to a partition (Z_+, Z_-, Z_o) is a **face** of this conical cell c, written $d \leq c$, if $Z_+ \subset Y_+$ and $Z_- \subset Y_-$. That is, the **face relation** $d \leq c$ holds if and only if some of the *equalities* defining d are converted to *inequalities* in the definition of c, while all *inequalities* defining d remain unchanged.

The **face at infinity** c_∞ of a conical cell c in X with vertex x is the set of ideal points $\xi \in X_\infty$ such that the *open* geodesic ray

$$(x, \xi) = [x, \xi) - \{x\}$$

lies inside c. An **ideal simplex** or **simplex at infinity** inside X_∞ is a subset σ of X_∞ so that there is *some* conical cell c in X so that

$$\sigma = c_\infty$$

Let c, d be two conical cells both with vertices at x. Say that the ideal simplex d_∞ is a *face* of the ideal simplex c_∞ if d is a face of c. We write $d_\infty \le c_\infty$ for this relation. This defines the **face relation** on ideal simplices. (We prove that it deserves this name in the theorem below).

Recall (4.6) that a thick building in which the apartments are Coxeter complexes $(\bar{W}, \bar{S})$ with $\bar{W}$ *finite* is said to be a **spherical building**.

For an apartment A in X, let A_∞ be the subset of X_∞ consisting of parallelism classes of geodesic rays with representatives in A (16.8). And we also think of A_∞ as *containing* the ideal simplices which are the faces at infinity of conical cells in A. Keep in mind that we are using the *maximal* apartment system (4.4) in the affine building X.

Theorem: The ideal simplices partition X_∞. The face relation is well-defined, and the poset of ideal simplices in X_∞ is a simplicial complex. Indeed, X_∞ is a *spherical building*, in the sense that the poset given by ideal simplices is a thick spherical building. Its apartments are in bijection with those in the *maximal* apartment system of the thick affine building X.

Remarks: Recall that there is a *unique* system of apartments in a spherical building (4.6).

Proof: This argument is broken into pieces, some of which are of minor interest in their own right, and may be of later use.

The following proposition generalizes the analogous fact for zero-dimensional ideal simplices, which was proven (in effect) in the previous section (16.8).

Proposition: Fix $x \in X$. Then the map $c \to c_\infty$ from conical cells with vertex x to ideal simplices is a *bijection*.

Proof: Let $\sigma = d_\infty$ be the face at infinity of the conical cell d with vertex y lying in an apartment B. Let $\mathcal{D}$ be a *sector* in B with vertex y so that d is a face of $\mathcal{D}$. By the discussion of configurations of chamber and sector, there is a subsector $\mathcal{D}'$ of $\mathcal{D}$ so that x (thought of as lying in the closure of some chamber) and $\mathcal{D}'$ lie in a common apartment A.

Now the subsector $\mathcal{D}'$ is a *translate* $\mathcal{D}' = t + \mathcal{D}$ of $\mathcal{D}$ (within the apartment B). And such translation preserves parallelism of geodesic rays. Thus, $d' = t + d$ is a face of $\mathcal{D}'$, and $t + d$ has the same face at infinity as does d.

By translating once more, this time inside the *other* apartment A, we can move d' to a conical sector in A with vertex x and with the same face at infinity.

The uniqueness follows from the definitions and from the uniqueness of *rays* with given direction and given vertex (16.8). ♣

Proposition: The ideal simplices are disjoint subsets of X_∞. Given two ideal simplices σ, τ, there is an apartment A in the maximal system in X so that there are two conical cells in A with faces at infinity σ, τ.

Proof: The second assertion will be proven incidentally in the course of proving the first.

Every ray with vertex x is contained in one of the conical cells with vertex x. Thus, every point in X_∞ lies inside *some* ideal simplex.

On the other hand, let σ, τ be distinct ideal simplices. Let c be a conical cell in an apartment A with vertex x whose face at infinity is σ, and let d be a conical cell in an apartment B with vertex y whose face at infinity is τ. Let $\mathcal{C}, \mathcal{D}$ be sectors in A, B of which c, d are faces.

There are subsectors $\mathcal{C}', \mathcal{D}'$ of $\mathcal{C}, \mathcal{D}$ which lie in a common apartment (16.7). We can write $\mathcal{C}' = u + \mathcal{C}$ for some translation u in A, and $\mathcal{D}' = v + \mathcal{D}$ for some translation v in B. Then $c' = u + c$ and $d' = v + d$ are conical cells in A, B with the same faces at infinity as c, d, and $u + c, v + d$ are faces of $\mathcal{C}', \mathcal{D}'$. Thus, c', d' lie in a common apartment. We can then translate them inside *that* apartment so that they have a common vertex. This certainly gives the second assertion of the proposition.

By the previous result, if σ, τ are distinct then so are c', d'. Thus, we have reduced the issue of disjointness to that of the disjointness of distinct conical cells. The latter is relatively elementary, and was discussed in detail in the discussion of reflection groups (12.1), (13.1). ♣

Proposition: Given two sectors $\mathcal{C}, \mathcal{D}$, we have $\mathcal{C}_\infty = \mathcal{D}_\infty$ if and only if $\mathcal{C}$ and $\mathcal{D}$ have a common subsector.

Proof: The sectors may be replaced by subsectors without changing their face at infinity, so may be taken to lie in a common apartment A, by the result on configuration of two sectors (16.7). Then we can write $\mathcal{C} = x + c$ and $\mathcal{D} = y + c$ for some conical cell c in A of maximal dimension. Changing signs of functionals if necessary, we may suppose that c is defined by a family of inequalities $\lambda > 0$. This family is *finite* since A is *affine*. Then any $z \in c$ with $\lambda z > \lambda x$ and $\lambda z > \lambda y$ lies in the intersection $\mathcal{C} \cap \mathcal{D}$. Thus, the intersection is a sector itself.

On the other hand, if two sectors *do* have a common subsector, it is easy to check that they have the same face at infinity. ♣

Now we can prove that X_∞ (or, really, the collection of ideal simplices) is a thick building, whose apartments are Coxeter complexes attached to *finite*, that is, *spherical* Coxeter groups. (Thus, we call X_∞ itself *spherical*).

Lemma: The set of ideal simplices in A_∞ is a finite Coxeter complex.

Proof: The collection of such simplices, together with face relations, is isomorphic as a poset to the finite Coxeter complex of conical cells with chosen vertex. That the latter is a Coxeter complex at all is a consequence of our study of reflection groups (12.2), (13.2). That it is *finite* is a consequence of the assumption that the apartment A is an *affine* Coxeter complex: the Perron-Frobenius computation shows this (13.3), (13.6). ♣

Lemma: The poset X_∞ (by which we really mean the poset of ideal simplices) is a simplicial complex.

Proof: We need to show two things (3.1). First, we show that for given ideal simplex σ the collection $(X_\infty)_{\leq\sigma}$ of all $\tau \leq \sigma$ is isomorphic to the set of subsets of a finite set. Second, we show that any two σ, τ in this poset have a greatest lower bound, that is, γ so that $\gamma \leq \sigma$ and $\gamma \leq \tau$ and so that if $\delta \leq \sigma$ and $\delta \leq \tau$ then $\delta \leq \gamma$.

A given σ *and all its faces* lie in some A_∞, which is a simplicial complex, so $(X_\infty)_{\leq\sigma} = (A_\infty)_{\leq\sigma}$ certainly is isomorphic as poset to the set of all subsets of a finite set.

And in a proposition just above we saw, in effect, that any two ideal simplices σ, τ lie in a common A_∞. Since the latter is a simplicial complex, all γ so that $\gamma \leq \sigma$ or $\gamma \leq \tau$ lie inside A_∞. Thus, since A_∞ is a simplicial complex, there is a greatest lower bound *inside* A_∞, which must also be the greatest lower bound inside X_∞. ♣

Corollary: Each A_∞ is a simplicial subcomplex of X_∞.

Proof: We already knew that A_∞ was a simplicial complex in its own right, so this corollary follows from the fact that we now know the whole building X_∞ to be a simplicial complex, invoking the criterion (3.1) for a poset to be a simplicial complex. ♣

And, the property that any two ideal simplices in X_∞ lie in a common apartment is one of the requirements for X_∞ to be a building with apartment system

$$\mathcal{A}_\infty = \{A_\infty : A \in \mathcal{A}\}$$

where $\mathcal{A}$ is the *maximal* apartment system in X (4.1).

Next, we must check the other axiom, that if two subcomplexes A_∞, B_∞ in $\mathcal{A}_\infty$ (obtained from apartments A, B in X) have a common chamber σ, then there is a chamber complex isomorphism $\phi : A_\infty \to B_\infty$ which is the identity map on $A_\infty \cap B_\infty$. Let $\mathcal{C}$ be a sector (maximal dimension conical cell) in A whose face at infinity is σ, and let $\mathcal{D}$ be a sector in B whose face at infinity is also σ. Just above, we saw that two sectors have the same face at infinity if and only if they have a common subsector. Thus, the existence of the common chamber requires there to be a common subsector $\mathcal{C}'$ of $\mathcal{C}$ and $\mathcal{D}$. Then, since X itself is a building, there is an isomorphism $\Phi : A \to B$ fixing $A \cap B$ (and the latter contains a sector $\mathcal{C}'$).

Since Φ (or its geometric realization, really) is an isometry, it must map parallelism classes of geodesic rays in A to such in B, so we obtain a natural map $\Phi_\infty : A_\infty \to B_\infty$.

We will show further that Φ_∞ fixes (the geometric realization of) $A_\infty \cap B_\infty$ pointwise. Fix $x \in A \cap B$, and let σ be a simplex in $A_\infty \cap B_\infty$. Then the set

$$x * \sigma = \bigcup_{\xi \in \sigma} (x, \xi)$$

(where (x, ξ) is the *open* geodesic ray) is the conical cell in A (or in B) with vertex x and face at infinity σ. Here we pointedly use the fact that the notion of geodesic is *intrinsic*, as was shown when the canonical metric on an affine building was first introduced (14.2).

In particular, $x * \sigma$ is contained in $A \cap B$, so Φ is trivial on $x * \sigma$. Thus, Φ_∞ is trivial on σ. This holds for any σ in $A \cap B$. This proves the second building axiom (in its stronger variant form (4.1)).

In particular, in the extreme case that $A_\infty = B_\infty$, the previous two paragraphs show that $x * \sigma$ for all σ in $A_\infty = B_\infty$. That is, all geodesic rays with vertex x inside A lie also in B, and vice-versa. Thus, $A = B$, and we have the asserted *bijection of apartments*.

Now we address the issue of *thickness*.

Given a simplex σ with vertex y, lying inside a prescribed apartment A', we define an associated conical cell c_σ by **extending σ inside A' from** y, in the following manner. Let c_σ be the union of all the open geodesic rays (y, ξ) inside A' emanating from y and which meet σ in a non-trivial geodesic line segment. Alternatively, the conical cell c_σ is the collection of all expressions $(1-t)y + tv$ for $t > 0$ and $v \in \sigma$.

Proposition: If σ, τ are distinct simplices both with vertex y, lying in apartments A_1, A_2, respectively, then the conical cells c_σ, c_τ obtained by extending σ, τ from y inside A_1, A_2 have distinct faces at infinity, regardless of choice of the apartments A_1, A_2.

Proof: Suppose that ξ were a common point of the two faces at infinity. Recalling the proposition of the previous section (16.8), there is a unique (open) geodesic (y, ξ) emanating from y and in direction ξ. Its intersection with a small enough neighborhood of y must lie inside *both* σ and τ. Thus, (the geometric realizations of) the simplices σ and τ have a common point. Since these are *open* simplices, it must be that $\sigma = \tau$. ♣

We need the fact, proven just above, that for *any* fixed point $x \in X$, every simplex in X_∞ occurs as the face at infinity of exactly one conical cell with vertex x.

Further, we use the fact proven earlier (12.4), (13.6) that in a given apartment $A = \Sigma(W, S)$ in an affine building X, there is at least one *good* (or *special* vertex x in the fundamental chamber $C = \langle \emptyset \rangle$). More specifically, under the natural surjection $W \to \bar{W}$ we have an isomorphism $W_x \to \bar{W}$, where W_x is the subgroup of W fixing x. And under this map S_x, the subset of S of reflections fixing x, is mapped surjectively to $\bar{S}$.

Since this was not emphasized earlier, note that x is *good* in any apartment B containing the chamber C in A of which x is a vertex. Indeed, by the building axioms there is an isomorphism $\phi : B \to A$ fixing C and x. That is, ϕ gives an isomorphism of these two Coxeter complexes, so any intrinsic property x has in one it will have in the other.

The following proposition illustrates the importance of special vertices: the fact that there exist special vertices implies that conical cells are geometric realizations of *simplicial* objects.

Proposition: Let c be a conical cell with vertex x, a special vertex. Let A be an apartment whose geometric realization $|A|$ contains c. Then there is a simplicial subcomplex τ of A whose geometric realization $|\tau|$ is c.

Proof: We use Tits' cone model (13.1), (13.5), (13.6) of the geometric realization of the affine Coxeter complex A. Choose a hyperplane η through x inside $|A|$ in each parallelism class, and let λ_η be a non-zero affine functional which is 0 on η.

As in (13.1), (13.5), the geometric realizations $|\tau|$ of simplices τ of which x is a vertex are described by all choices of equalities $\lambda_\eta(y) = 0$ or inequalities $\lambda_\eta(y) > 0$ or $\lambda_\eta(y) < 0$ as η ranges over hyperplanes through x, *together with* an additional inequality $\lambda_o(y) \geq 0$, where λ_o is a non-zero affine functional vanishing on the opposite facet to x in some chamber in A of which x is a vertex.

If the latter condition $\lambda_o(y) \geq 0$ defining $|\tau|$ is *dropped*, then we obtain the conical cell *extending* the simplex τ from x inside A, in the sense above.

On the other hand, suppose we are given a conical cell c with vertex x. By definition, c is described by some inequalities and equalities employing all the functionals λ_η. If all inequalities are changed to *strict* inequalities, and equalities $\lambda_\eta(y) = 0$ changed to strict inequalities $\lambda_\eta(y) > 0$, then the subset $\mathcal{C}'$ of $|A|$ so defined is non-empty (13.1), being a chamber for the spherical Coxeter group W_x in Tits' cone model.

Since the chambers cut out by the *whole* affine Coxeter group are literal simplices, there must be some other hyperplane η_o which cuts $\mathcal{C}'$ into two pieces, one of which is a literal simplex $|C'|$ for some chamber C' in A. Let λ_o be a non-zero affine functional which vanishes on η_o and is positive on C'.

Now change all the strict inequalities back to their original forms which defined c, but adjoin the inequality $\lambda_o(y) > 0$. The set $|tau|$ so defined is the geometric realization of a face τ of C' (13.1), (13.5), (13.6).

Thus, when a special vertex is used as vertex for conical cells, the conical cells are geometric realizations of simplicial subcomplexes of the ambient apartment. ♣

Returning to the proof of the theorem: Let d be a codimension one conical cell with vertex at the special vertex x, whose face at infinity is therefore a facet F_∞ in X_∞. Since x is *good*, d contains a facet (codimension one simplex) F with vertex x in X. Since X is thick, there are at least three chambers C_1, C_2, C_3 in X with facet F.

Invoking the proposition a little above, we see that these three chambers give rise to sectors with distinct faces at infinity (possibly in a variety of ways). Thus, X_∞ with the apartment system $\mathcal{A}_\infty$ is a thick spherical building.

Thus, the theorem is proven. ♣

Remarks: It may be observed that the previous discussion blurs somewhat the distinction between the spherical building at infinity and its geometric realization, and between simplicial complex maps and their geometric realizations. Indeed, the collection of *points at infinity*, which is the geometric realization, was constructed *first*. Yet in the end the *faces at infinity* of conical cells, as subsets of the collection of *points at infinity*, and with the *face relations* inherited from the conical cells, really does constitute a poset which is the desired simplicial complex.

Remarks: It is not difficult to investigate the situation wherein the apartment system $\mathcal{A}$ in the affine building is *not* maximal. The bijection of apartments proven above, with the fact that spherical buildings have unique apartment systems, is an indicator that the *building* at infinity itself, not merely its apartment system, must be smaller to accommodate this. Indeed, the only hope is to take

$$X_{\infty,\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A_\infty$$

with apartment system

$$\mathcal{A}_\infty = \{A_\infty : A \in \mathcal{A}\}$$

with A_∞ the subcomplex of X_∞ as above. Yet this $X_{\infty,\mathcal{A}}$ will not satisfy the building axioms unless we *further* explicitly require of $\mathcal{A}$ that *any two sectors in X have subsectors which lie in a common apartment in $\mathcal{A}$*. But for applications to p-adic groups there is scant reason to consider any other than the maximal system.

16.10 Induced maps at infinity

Not surprisingly, in broad terms, automorphisms of a thick affine building give rise to automorphisms of the associated thick spherical building at infinity. This section makes the idea precise. An important corollary at the end compares the stabilizer of an apartment in the affine building with the stabilizer of the corresponding apartment in the spherical building.

Proposition: If ϕ is an *isometry* of the geometric realization $|X|$ of the thick affine building X, then ϕ preserves parallelism classes of geodesic rays, so it gives a well-defined map ϕ_∞ on the geometric realization $|X_\infty|$ of the building at infinity, by

$$\phi([x,\xi)) = [\phi x, \phi_\infty \xi)$$

where ξ is a point at infinity and x is *any* point in $|X|$.

Proof: Let

$$\gamma, \delta : [0,\infty) \to |X|$$

be two geodesics in a parallelism class $\xi \in |X_\infty|$, as above (16.8). That is, these maps are isometries, and the supremums

$$\sup_s \sup_t \; d(\gamma s, \delta t)$$

$$\sup_t \sup_s \; d(\gamma s, \delta t)$$

are both finite. Having thus unraveled the definition, it is immediate that an isometry preserves this property. The notational style of the assertion of the proposition is merely a paraphrase of this. ♣

But the map ϕ_∞ does *not* directly give a *simplicial complex map* on X_∞. The following theorem and its corollary address the simplicial complex issue, including *labeling*.

Theorem: Let $f : X \to X$ be a simplicial-complex automorphism of the thick affine building X, with its maximal apartment system. Then the geometric realization $|f|$ of f maps *conical cells* to conical cells in $|X|$, and defines a simplicial-complex automorphism f_∞ of X_∞ by

$$f_\infty(c_\infty) = (fc)_\infty$$

where c is a conical cell and c_∞ is its face at infinity. If f is label-preserving, then so is the induced map f_∞.

Proof: First, as in the discussion of labels, links, and maximal apartment system, we know that there is a *unique* maximal apartment system $\mathcal{A}$ (4.4). Since the collection $f\mathcal{A}$ of images fA for $A \in \mathcal{A}$ is certainly another apartment system, inevitably $f\mathcal{A} = \mathcal{A}$. Thus, f maps apartments to apartments.

From the discussion of the canonical metric on affine Coxeter complexes (13.7), since f gives a simplicial complex isomorphism $A \to fA$ on apartments A, the geometric realization $|f|$ of f is an *isometry* from $|A|$ to $|f(A)|$. By the building axioms (4.1) any pair of points in $|X|$ is contained in a common apartment, so $|f|$ is an *isometry* on the whole building. Thus, by the previous little proposition, $|f|$ gives a well-defined map on *points* in $|X_\infty|$.

And f certainly maps walls in A to walls in $f(A)$, since apartments are Coxeter complexes, and since every pair of adjacent chambers in a Coxeter complex is separated by a wall (3.6). Therefore, from the definition of conical cells (16.9), the geometric realization $|f|$ of f preserves the collection of *conical cells* in $|X|$.

Further, since f is a simplicial complex map, it preserves the face relations among conical cells.

Thus, we can attempt to define f_∞ on X_∞, by

$$f_\infty(c_\infty) = (f(c))_\infty$$

If this map is well-defined, then we have what we want.

Since $|f|$ has been shown to preserve parallelism classes of geodesic rays, we already have a partial result in the direction of well-definedness: for a conical cell c in an apartment A, and for a translation t inside A,

$$f_\infty(c_\infty) = f_\infty((t+c)_\infty)$$

Indeed, the geodesic rays in $t+c$ are visibly parallel to corresponding rays in c, and parallelism is respected by $|f|$.

Now treat the general case: the argument recapitulates some ideas used just above. Given two conical cells c, d with the same face at infinity, we choose sectors $\mathcal{C}, \mathcal{D}$ of which c, d are faces. Let A, B be apartments containing $\mathcal{C}, \mathcal{D}$, respectively. From above (16.7), there are subsectors $\mathcal{C}', \mathcal{D}'$ of $\mathcal{C}, \mathcal{D}$ (respectively) which lie in a common apartment A'. Writing $\mathcal{C}' = \mathcal{C} + u$ and $\mathcal{D}' = \mathcal{D} + v$ for some translations u, v in A, B, respectively, we have conical cells $c+u$ and $d+v$ which are translates (in A, B respectively) of c, d and now lie in a common apartment A'. Finally, we translate $(c+u)$ inside A' to arrange that the two conical cells have the same vertex: let $(c+u)+w$ be this translate. (The extreme ambiguity of notation here is harmless).

In the previous section (16.9) it was shown that, for given vertex in $|X|$ there is a unique conical cell having that vertex and having prescribed face at infinity. Thus, in the present situation, it must be that

$$(c+u)+w = d+v$$

Note that translation (in any apartment) does not change parallelism classes of geodesic rays, so does not change faces at infinity.

Since we have noted that $|f|$ respects parallelism, we can compute:

$$f_\infty(c_\infty) = (|f|c)_\infty = (|f|(c+u))_\infty = (|f|((c+u)+w))_\infty$$

Since $(c+u)+w = d+v$, this is the same as

$$(|f|(d+v))_\infty = (|f|d)_\infty = f_\infty(d_\infty)$$

This proves the well-definedness.

It remains to check that labels in X_∞ are preserved by f_∞.

Recall that buildings and Coxeter complexes both are *uniquely* labelable (up to isomorphism of labelings), and that the maps required to exist by the building axioms are all label-preserving (4.4). This is as explicit as we need to be about the labeling.

Consider first the easy case that A and fA have a sector $\mathcal{C}$ in common, and that $f : A \to fA$ is the isomorphism ϕ fixing $A \cap fA$ as required by the building axioms. Then A_∞ and $(fA)_\infty$ have the common chamber $\mathcal{C}_\infty$. Let

ϕ be the isomorphism $A_\infty \to (fA)_\infty$ from the building axioms. As noted earlier, this isomorphism preserves labels (4.4).

On the other hand, from the Uniqueness Lemma (3.2) it is easy to see that there is a *unique* simplicial-complex isomorphism $A_\infty \to (fA)_\infty$ trivial on the chamber $\mathcal{C}_\infty$. Since $f\mathcal{C} = \mathcal{C}$, by definition $f_\infty \sigma_\infty = \sigma_\infty$ for every conical cell σ which is a face of $\mathcal{C}$, so f_∞ has this property. Therefore, it must be that $f_\infty = \phi$, so f_∞ preserves labels in this easy case.

In the general case, given A and fA, let $\mathcal{C}, \mathcal{D}$ be sectors in these apartments, respectively. Shrink these sectors to subsectors small enough so that without loss of generality both $\mathcal{C}, \mathcal{D}$ lie in a common apartment B (16.7). Let $\phi : A \to B$ and $\psi : B \to fA$ be the isomorphisms trivial on $A \cap B$ and $B \cap fA$, respectively, as postulated by the axioms. Let Φ be the composite $\psi \circ \phi$. By the *easy case* just treated,

$$\Phi_\infty = (\psi_\infty) \circ (\phi_\infty)$$

is label-preserving on X_∞.

The composite $\Phi^{-1} \circ f$ on X thus gives a label-preserving simplicial-complex automorphism of the Coxeter complex A. Choosing an identification of A with a literal Coxeter complex $\Sigma(W, S)$, there is $w \in W$ so that the restriction of $\Phi^{-1} \circ f$ to A is just multiplication by w.

Fix a *special vertex* x in A, and suppose that $|A|$ is given a real vectorspace structure with $x = 0$. Identify the simplicial complex A_∞ with the collection of conical cells with vertex x. For $w \in W$, write $w = \bar{w} w_T$ where w_T is the translation part of w and $\bar{w}$ is the linear part (12.4), (13.6). Since the translation part certainly preserves parallelism classes, w_T acts *trivially* on faces at infinity. Thus, the induced action of $w \in W$ on the faces at infinity of such conical cells in A is just by its *linear part* $\bar{w}$ lying inside the *finite* Coxeter group $\bar{W}$.

Thus, by construction of A_∞ in terms of conical cells, $\bar{W}$ is likewise identified with the label-preserving simplicial-complex automorphisms of the apartment A_∞ inside X_∞. Thus,

$$f_\infty = \Phi_\infty \circ \bar{w}$$

is label-preserving, as desired. ♣

Finally, we have

Corollary: Let f be a simplicial-complex automorphism of the thick affine building X. Then f stabilizes the apartment A in X if and only if the map induced by f on the spherical building X_∞ stabilizes the corresponding apartment A_∞ at infinity.

Proof: The apartment A_∞ is the collection of simplices in X_∞ obtained as faces at infinity of conical cells in A. And the conical cells with vertex a *special* vertex x are geometric realizations of simplicial subcomplexes of the apartment. So if f stabilizes A it certainly stabilizes A_∞.

The other containment is non-trivial. Given a chamber D in A, let y be a point and $y+c$ a sector so that $D \subset y+c$. Choose a point $z \in y+c$ so that $D \subset z-c$. Then, as proven in the discussion of sectors in affine Coxeter complexes (16.1), for any two chambers $C_1 \subset z+c$ and $C_2 \subset y-c$ the chamber D occurs in *some* minimal chamber γ from C_1 to C_2.

Let f stabilize A_∞. Then for any sector $\mathcal{D}$ in A its image $f\mathcal{D}$ contains a subsector lying in A, by the definition of A_∞ and by the definition of induced maps at infinity. For given f, choose C_1, C_2 in $y+c, z-c$ so that fC_1 and fC_2 both lie in A. Then $f\gamma$ is still a minimal gallery connecting fC_1, fC_2, and containing fD. By the *combinatorial convexity* of apartments in *any* thick building (4.5), it follows that $f\gamma$ and hence fD lie inside A.

That is, if f stabilizes A_∞ then f stabilizes the collection of *chambers* in A, so (being a simplicial complex map) f necessarily stabilizes the apartment A. ♣

17. Applications to Groups

- Induced group actions at infinity
- BN-pairs, parahorics and parabolics
- Translations and Levi components
- Filtration by sectors: Levi decompositions
- Bruhat and Cartan decompositions
- Iwasawa decompositions
- Maximally strong transitivity
- Canonical translations

Now consider a group G acting on a thick *affine* building X, so that the subgroup G^o of G *preserving labels* is *strongly transitive*. (We will be concerned almost entirely with just the *maximal* apartment system). (Earlier (5.5), when we talked about generalized BN-pairs, we used a different notation: $\tilde{G}$ was the large group and G the label-preserving subgroup).

This situation gives rise to a (strict) BN-pair in G^o, and to a *generalized* BN-pair in G (5.5). These are the **affine** BN-pairs in G^o and G. The *spherical building at infinity* yields *spherical* BN-pairs in G^o and in G. A new and important thing is the *interaction* of the affine and spherical BN-pairs.

17.1 Induced group actions at infinity

The point here is to show that a good group action on a thick affine building gives rise to a reasonable group action on the (thick) spherical building at infinity.

Let G be a group acting on a thick affine building X by simplicial-complex automorphisms. Suppose that the subgroup G^o of G acting by *label-preserving* automorphisms is *strongly transitive*, in the usual sense that it is transitive on pairs (C, A) where C is a chamber contained in an apartment A, where A lies in the *maximal* apartment system. Here and in the sequel we only consider the *maximal* apartment system (4.4) in X, and (unavoidably) the *unique* apartment system (4.6) in the spherical building X_∞.

Theorem: Under the induced maps, G^o acts *strongly transitively* on the thick spherical building X_∞ at infinity, and *preserves labels*.

Proof: This is mostly a corollary of prior results (16.9) about the spherical building at infinity, and about induced maps on the building at infinity (16.10), together with a review of more elementary facts.

From the uniqueness of the *maximal* apartment system (4.4) it follows that G unavoidably stabilizes the set $\mathcal{A}$ of apartments in X. From the discussion of induced maps at infinity (16.10), elements of G induce simplicial-complex automorphisms of X_∞. Further, we have already shown (16.10) that label-preserving maps on X induce label-preserving maps on X_∞. Thus, the issue is the transitivity of G^o on pairs (C_∞, A_∞) where C_∞ is a chamber in the apartment A_∞ in X_∞.

By the main theorem on the building at infinity (16.9), the apartments A_∞ are in bijection with the apartments A in the maximal system $\mathcal{A}$ in X, so the transitivity of G^o on $\mathcal{A}$ gives transitivity on the system $\mathcal{A}_\infty$ in X_∞. And, again, the fact that G^o stabilizes the set of apartments in X_∞ *follows* from the fact that that apartment system is unavoidably the *maximal* one (being unique (4.6), since the building at infinity is spherical).

Let $\Sigma(W, S)$ be a Coxeter complex isomorphic to apartments A in X (4.3). (From the discussion of *links* they are all isomorphic (4.4)). Let x be a *special vertex* in A (12.4), (13.6), and give $|A|$ a real vectorspace structure so that $x = 0$. We have shown that the faces at infinity of conical cells in $|A|$ are in bijection with conical cells with vertices at x (16.9).

Every $w \in W$ can be written as $w = \bar{w}w_T$ where $\bar{w}$ is the linear part of w and w_T is the translation part. This is essentially the definition of specialness of the vertex $x = 0$. Translations do not move geodesic rays out of their parallelism classes, so faces at infinity are not altered by w_T. Thus, the only effect w has on faces at infinity is by $\bar{w}$.

Then the image complex $(\bar{W}, \bar{S})$ under the map $w \to \bar{w}$ is the *finite* Coxeter system whose associated complex gives the isomorphism class of the apartments at infinity (12.4), (13.3), (13.6). For any choice of isomorphism $A \approx \Sigma(W, S)$ we obtain an identification of $W = W_{\text{aff}}$ with the label-preserving automorphisms of A, and of $\bar{W} = W_{\text{sph}}$ with the label-preserving automorphisms of A_∞.

Let $\mathcal{N}^o$ be the stabilizer in G^o of a fixed apartment A in X. By hypothesis, $\mathcal{N}^o$ is transitive on chambers in A. From the Uniqueness Lemma (3.2), a label-preserving automorphism of a Coxeter complex is determined completely by what it does to one chamber. Thus, as noted already in the basic discussion of BN-pairs (5.2), the natural map $\mathcal{N}^o \to W$ is a *surjection*.

The action of $\mathcal{N}^o$ on chambers in A_∞ is by way of the composite

$$\mathcal{N}^o \to W \to \bar{W} = W_{\text{sph}}$$

so is *transitive* on chambers in the given apartment, as claimed. ♣

Let $C_\infty \subset A_\infty$ be a choice of chamber in an apartment in the associated spherical building X_∞ at infinity. Let

$$\mathcal{N}^o_{\text{sph}} = \text{ stabilizer in } G^o \text{ of the apartment } A_\infty$$

$$B^o_{\text{sph}} = \text{ stabilizer in } G \text{ of the chamber } C_\infty$$

Corollary: The pair $\mathcal{N}^o_{\text{sph}}$, B^o_{sph} is a (strict) spherical BN-pair. ♣

17.2 BN-pairs, parahorics and parabolics

Throughout this section we continue to suppose that G is a group acting on a thick affine building X with the hypothesis that the label-preserving subgroup G^o of G is strongly transitive (with respect to the maximal apartment system). We will begin to see what things can be said about the group in terms of the 'obvious' geometry of the affine building and the spherical building at infinity.

This section sets up notation which will be used throughout the rest of this chapter.

Fix a chamber C in an apartment A in X, let x be a vertex of C which is *special* (14.8), and fix a sector $\mathcal{C}$ inside A with vertex x and containing C (16.9). Let C_∞ be the face at infinity of $\mathcal{C}$ and let A_∞ be the apartment at infinity consisting of all faces at infinity of conical cells inside A (16.9).

Let

$$\begin{array}{lcl}
\mathcal{N}_{\text{aff}} & = & \text{stabilizer in } G \text{ of the apartment } A \\
B & = & \text{stabilizer in } G \text{ of the chamber } C \\
T & = & \mathcal{N} \cap B \\
P & = & \text{stabilizer in } G \text{ of the chamber } C_\infty \\
\mathcal{N}_{\text{sph}} & = & \text{stabilizer in } G \text{ of the apartment } A_\infty \\
M & = & \mathcal{N}_{\text{sph}} \cap P
\end{array}$$

$$\begin{array}{lcl}
\mathcal{N}^o_{\text{aff}} & = & \text{stabilizer in } G^o \text{ of the apartment } A \\
B^o & = & \text{pointwise fixer in } G^o \text{ of the chamber } C \\
T^o & = & B \cap \mathcal{N}^o = \text{ pointwise fixer in } G^o \text{ of the apartment } A \\
P^o & = & \text{stabilizer in } G^o \text{ of the chamber } C_\infty \\
\mathcal{N}^o_{\text{sph}} & = & \text{stabilizer in } G^o \text{ of the apartment } A_\infty \\
M^o & = & \mathcal{N}^o_{\text{sph}} \cap P^o = \text{ pointwise fixer in } G^o \text{ of the apartment } A_\infty
\end{array}$$

Recall that T normalizes $\mathcal{N}^o$, B^o, and G^o, and that T and G^o together generate G (5.5). Thus, G^o is a normal subgroup of G, and is of finite index (5.5) since the building X is finite-dimensional. Let

$$\Omega = G/G^o \approx \mathcal{N}_{\text{aff}}/\mathcal{N}^o_{\text{aff}} \approx T/T^o$$

be the quotient.

The **Weyl groups** are

$$\begin{array}{lcl}
W & = & W_{\text{aff}} = \text{ affine Weyl group } = \mathcal{N}^o/(\mathcal{N}^o \cap B^o) \\
\bar{W} & = & W_{\text{sph}} = \text{ spherical Weyl group } \approx \mathcal{N}^o/\mathcal{N}^o_{\text{trans}}
\end{array}$$

where $\mathcal{N}^o_{\text{trans}}$ is the subgroup of $\mathcal{N}^o$ consisting of elements whose restrictions to A are translations. Also, by definition, for a special vertex x in A

$$W_{\text{sph}} \approx W_x$$

where W_x is the subgroup of W fixing x.

From the demonstrated strongly transitive action on the spherical building at infinity (17.1), we also have a *strict spherical* BN-pair $P^o, \mathcal{N}^o_{\mathrm{sph}}$ and a *generalized spherical* BN-pair $P, \mathcal{N}_{\mathrm{sph}}$.

Note: While we are assured that the action at infinity of G^o is label-preserving (16.10), it is not clear how much *larger* than G^o the subgroup of *G preserving labels at infinity* might be. In some cases, the whole group G preserves labels *at infinity*, but there are natural examples where this is not so.

Note that, to distinguish the two cases, The 'B' in the spherical case will be denoted P and P^o (in G and G^o, respectively), and called a **minimal parabolic** or **Borel** subgroup, while the 'B' in the affine case is denoted B and B^o (in G and G^o, respectively), and will be called an **Iwahori subgroup**. The subgroups M and M^o inside P and P^o are *Levi components* of P and P^o (respectively).

Any subgroup of G containing B^o is called a **parahoric subgroup** of G. Any subgroup of G containing P is called a **parabolic** subgroup of G.

Apart from setting up notation, the point of this section is to note that the 'N' *is the same* in both the affine and spherical BN-pairs:

Theorem: We have

$$\mathcal{N}_{\mathrm{aff}} = \mathcal{N}_{\mathrm{sph}}$$
$$\mathcal{N}^o_{\mathrm{aff}} = \mathcal{N}^o_{\mathrm{sph}}$$

Proof: This is the obvious corollary of the fact that a simplicial complex automorphism of X stabilizes A if and only if it stabilizes A_∞ (16.10). ♣

Therefore, we write simply

$$\mathcal{N} = \mathcal{N}_{\mathrm{aff}} = \mathcal{N}_{\mathrm{sph}}$$
$$\mathcal{N}^o = \mathcal{N}^o_{\mathrm{aff}} = \mathcal{N}^o_{\mathrm{sph}}$$

Remarks: It is not generally true that the induced maps given by G^o constitute exactly the label-preserving subgroup of the group of maps induced by G on X_∞. To the contrary, in many natural examples *all* induced maps from G on X_∞ are label-preserving.

And the usual terminology is

$$W_{\mathrm{aff}} = \mathcal{N}^o/T^o = \text{ affine Weyl group}$$
$$W_{\mathrm{sph}} = \mathcal{N}^o/M^o = \text{ spherical Weyl group}$$

Note that these Weyl groups are defined in terms of the type-preserving group G^o rather than the whole group G. The fact that the type-preserving subgroup *at infinity* may be larger than G^o is irrelevant to determination of the spherical Weyl group, since the strong transitivity of G^o at infinity follows from that on the affine building (17.1). And the isomorphism class of the apartments at infinity is uniquely determined (5.6).

17.3 Translations and Levi components

Keep the notation from the previous section.

For this section, we suppose that G *preserves labels*, so in previous notation $G = G^o$, $B = B^o$, $\mathcal{N} = \mathcal{N}^o$, and so on. Let $\mathcal{N}_{\text{trans}}$ be the subgroup of$\mathcal{N}$ consisting of those group elements whose restriction to the apartment A are *translations* of A.

Let A_∞ be the apartment at infinity corresponding to the apartment A (16.9). Let x be a special vertex of the chamber C in A whose stabilizer is B. Let $\mathcal{C}$ be the sector in A with vertex x and containing C. Let C_∞ be the face at infinity of $\mathcal{C}$.

Theorem: Assuming that G preserves labels, the Levi component $M = \mathcal{N} \cap P$ of the minimal parabolic P in G with respect to the apartment A_∞ is the subgroup of *translations*

$$M = \mathcal{N}_{\text{trans}}$$

in M.

Proof: We use the fact that $\mathcal{N}$ is the 'N' in both the affine and spherical BN-pairs (17.2), and similarly for $\mathcal{N}$ in the generalized BN-pairs.

On one hand, we must show that $\mathcal{N}_{\text{trans}} \subset P$. Since $\mathcal{N}_{\text{trans}}$ acts on A by translations, the action of $\mathcal{N}_{\text{trans}}$ preserves parallelism, so preserves faces at infinity of conical cells. Thus, $\mathcal{N}_{\text{trans}} \subset P$.

On the other hand, if $g \in P$ then $g\mathcal{C}$ has a subsector in common with $\mathcal{C}$ (16.9), and if also $g \in \mathcal{N}$, then the image $g\mathcal{C}$ lies entirely within A. The image gx of the vertex x of $\mathcal{C}$ has the same label as does x, since $g \in G$, and gx is a vertex of gC.

Since $\mathcal{N}$ preserves labels and is transitive on chambers in A, it is necessarily transitive on pairs (x', C') where x' is a special vertex with the same label as x and C' is a chamber of which x' is a vertex. Thus, there is a unique $w \in W_{\text{aff}}$ so that $wC = gC$ and $wx = gx$. Let W_x be the subgroup of W_{aff} fixing the special vertex x. Since the composite map

$$W_x \subset W \to W_{\text{sph}}$$

is an isomorphism, we can write $w = w_1 w_x$ with $w_x \in W_x$ and w_1 a *translation* in W_{aff}. Then $w_1 x = gx$. Such translations w_1 preserve parallelism in A, so $w_1\mathcal{C} = g\mathcal{C}$. Thus, $g^{-1}w_1$ stabilizes the apartment A, stabilizes the sector $\mathcal{C}$, and fixes its vertex x. By the uniqueness lemma (3.2), $g^{-1}w_1$ acts trivially on A, as desired. ♣

Remarks: The analogous assertion for a non-necessarily label-preserving group G is not as simple as this. One half the argument still works, namely, that

$$\mathcal{N}_{\text{trans}} \subset \mathcal{N} \cap P$$

(where $\mathcal{N}_{\text{trans}}$ is the subgroup of $\mathcal{N}$ of elements whose restrictions to A are translations). However, in general this containment is *strict*.

17.4 Filtration by sectors: Levi decomposition

Under the hypothesis that the group G preserves labels, there is a decomposition result for minimal parabolics P

$$P = M \cdot N^{\#}$$

where $M = \mathcal{N}_{\text{trans}}$ is the subgroup of the stabilizer $\mathcal{N}$ of the chosen apartment A containing the chamber C of which P is the stabilizer. In the previous section this subgroup M was identified with a Levi component of P.

The subgroup $N^{\#}$ will be shown to be a *normal* subgroup of P, and is a 'thickened' form of the *unipotent radical* (often denoted 'N') of P: see (7.1),(7.4), (8.1-4) for descriptions for the classical groups. If it were *exactly* the unipotent radical then this decomposition would be the standard p-adic Levi decomposition.

The description of this $N^{\#}$ in terms of the *affine* building is immediately useful in at least one way: for the classical groups this will make it easy to verify that the spherical building at infinity is the same as the spherical building constructed directly earlier. In broader terms, the fact that such a description is possible in *this* context (as opposed to a more Lie-theoretic scenario) bodes well for the general utility of our approach.

More generally, let S_x be the reflections in A fixing the vertex x, let $S' \subset S_x$ and let $c^{S'}$ be the conical cell with vertex x extending the face $F^{S'}$ of type S' of the chamber C. We have the corresponding *parahoric subgroup*

$$B_{S'} = B\,S'\,B = \text{ pointwise fixer of the face of } C \text{ fixed by } S'$$

and *parabolic subgroup*

$$P_{S'} = P\,S'\,P = \text{ pointwise fixer of the face of } C_\infty \text{ fixed by } S'$$

Proposition: Assume that G preserves labels. The intersection $B \cap P$ is the *pointwise fixer* of the whole sector $\mathcal{C}$.

Proof: On one hand, if g fixes pointwise fixes a sector $\mathcal{C}$ with vertex x and containing the chamber C, then it certainly fixes C, and also fixes the face at infinity $\mathcal{C}_\infty$ of $\mathcal{C}$. That is, $B \cap P$ is contained in the pointwise fixer of the sector $\mathcal{C}$.

On the other hand, if g is in $B \cap P$ then it fixes C and face at infinity $\mathcal{C}_\infty$. Every chamber at infinity is the face at infinity of a unique sector with vertex x (16.9). Thus, if g stabilizes C (necessarily pointwise, since it is *label-preserving*), then $g\mathcal{C}$ is another sector with vertex x, since g is a simplicial automorphism of X and engenders an isometry on $|X|$ (13.7). On the other hand, g stabilizes $\mathcal{C}_\infty$, so $g\mathcal{C}$ must be $\mathcal{C}$. ♣

Now we can describe a subgroup $N^\#$ of the minimal parabolic P which is nearly the *unipotent radical* of P (7.1),(7.4), (8.1-4). As usual, let $M = \mathcal{N}_{\text{trans}}$ be the subgroup of G consisting of elements which stabilize the fixed apartment A and induce *translations* on A. Let C_∞ be the fixed chamber in the associated apartment at infinity A_∞. For any sector $\mathcal{D}$ (in any apartment) with face at infinity being the fixed chamber C_∞, let

$$N^o_{\mathcal{D}} = \text{ pointwise fixer of } \mathcal{D}$$

Then define

$$N^\# = \bigcup_{\mathcal{D}} N^o_{\mathcal{D}}$$

Proposition: This set $N^\#$ is a *subgroup* of G. It is equal to

$$N^\# = \bigcap_{\mathcal{D}\subset A} N_{\mathcal{D}}$$

and is normalized by M.

Proof: That $N^\#$ contains the identity and is closed under inverses is clear. From (16.9), two sectors $\mathcal{D},\mathcal{D}'$ have a common face at infinity if and only if they have a common subsector $\mathcal{D}$". Thus, for g fixing $\mathcal{D}$ and g' fixing $\mathcal{D}'$, the product gg' surely fixes $\mathcal{D}$". That is, $N^\#$ is a sub*group*.

Further, again from (16.9), every sector with face at infinity being the specified C_∞ has a subsector lying inside A. This proves the second assertion.

The subgroup M of the stabilizer $\mathcal{N}$ of A consisting of translations certainly maps sectors $\mathcal{D}$ to sectors $\mathcal{D}'$ having a common subsector with $\mathcal{D}$, so M fixes C_∞. Given $n \in N^\#$, let $\mathcal{D}$ be a subsector of A fixed by n, invoking the earlier part of this proposition. Then for $m \in M$ the element mnm^1 of G certainly stabilizes the sector $m^{-1}\mathcal{D}$ inside A. This sector still has face at infinity C_∞, so we have proven that M normalizes $N^\#$. ♣

Theorem: Assume that G preserves labels on the affine building. We have the decomposition

$$P = M \cdot N^\#$$

and $N^\#$ is *normal* in P .

Proof: On one hand, by its definition, $N^\#$ also fixes C_∞. Thus, $M \cdot N^\# \subset P$. This is the easy direction of containment.

On the other hand, by the strong transitivity, the subgroup $\mathcal{N}$ which stabilizes the apartment A is transitive on chambers inside A. Since $\mathcal{N}$ preserves labels and is transitive on chambers in A, it is necessarily transitive on pairs (x', C') where x' is a special vertex with the same label as x and C' is a chamber of which x' is a vertex. Thus, there is a unique $w \in W_{\text{aff}}$ so that $wC = gC$ and $wx = gx$.

Let $p \in P$. Then $p\mathcal{C}$ still has the same face at infinity, so has a common subsector $\mathcal{C}'$ with $\mathcal{C}$, by (16.9). Without loss of generality, $\mathcal{C}'$ has vertex a special vertex x' (which need not be of the same type as x). Let

$$\mathcal{C}_1 = p^{-1}\mathcal{C}' \subset p^{-1}(p\mathcal{C} \cap \mathcal{C}) \subset A$$

This has vertex x_1, which is necessarily a special vertex. Then

$$p\mathcal{C}_1 \subset A$$

and its vertex px_1 is a special vertex in A of the same type as x_1. Let $w \in \mathcal{N}$ be such that $wx_1 = px_1$. By the definition of *special*, the affine Weyl group W is a semi-direct product

$$W = W_{x_1} \cdot M = M \cdot W_{x_1}$$

where W_{x_1} is the subgroup of W fixing x_1. Thus, there is $m \in M$ so that $mx_1 = px_1$.

Therefore, we find that $m^{-1}p$ fixes x_1 and stabilizes the chamber C_∞ at infinity. From (16.9), there is a unique sector with vertex x_1 with face at infinity C_∞, which must be $\mathcal{C}_1$. Thus, $m^{-1}p \in N^\#$.

Since we have already seen that M normalizes $N^\#$, it now follows that $N^\#$ is a normal subgroup of P. ♣

17.5 Bruhat and Cartan decompositions

Keep the notation from above.

For the sake of completeness of the present line of discussion, we recall here the simplest parts of the Bruhat-Tits decomposition results as applied to both the affine and spherical BN-pairs.

Assuming that G preserves labels on the building at infinity, the traditional **Bruhat decomposition** (5.1) is

$$G = \bigsqcup_{\bar{w} \in W_{\mathrm{sph}}} P\bar{w}P$$

Again, let

$$\Omega = T/T^o = (\mathcal{N} \cap B)/(\mathcal{N}^o \cap B^o)$$

be as earlier. The **Cartan decomposition**, another example of a Bruhat-Tits decomposition (5.5), is

$$G = \bigsqcup_{w \in W_{\mathrm{aff}} \quad \sigma \in \Omega} B^o w \sigma B^o$$

17.6 Iwasawa decomposition

The Iwasawa decomposition is not simply a Bruhat-Tits decomposition, spherical or affine. Indeed, the very statements refer simultaneously to *parabolics* and *parahorics*: the interaction of the affine building and the spherical building at infinity play a significant role in the proof. We keep the notation from just above.

Theorem:

$$G^o = \bigsqcup_{\bar{w} \in W_{\mathrm{sph}}} P^o \bar{w} B^o$$

and

$$G^o = \bigsqcup_{\bar{w} \in W_{\mathrm{sph}} \quad \sigma \in \Omega} P^o \bar{w} \sigma B^o$$

Proof: We have shown that there is a subsector $g\mathcal{C}_1$ of $g\mathcal{C}$ (with $\mathcal{C}_1$ a subsectorof $\mathcal{C}$) so that both $g\mathcal{C}_1$ and the chamber C lie in a common apartment A_1 (16.5). The strong transitivity of G^o on X assures that B^o itself is *transitive* on apartments containing C. Thus, there is $b \in B^o$ so that $bA_1 = A$, so $bg\mathcal{C}_1 \subset A$.

Recall that a group H is said to act *simply transitively* on a set Θ if, for any $\theta \in \Theta$, $h\theta = \theta$ implies $h = 1$. (If this property holds for a single $\theta \in \Theta$, then it holds for *every* element of Θ).

Since W_{sph} is simply transitive on chambers in the Coxeter complex A_∞, it must be that W_{sph} is simply transitive on *parallelism classes of sectors* in A, where for sectors *parallel* means possessing a common subsector (16.9). Thus, there is a unique $\bar{w}$ in W_{sph} so that $\bar{w}bg\mathcal{C}_1$ has a subsector in common with $\mathcal{C}$.

Then the larger sector $\bar{w}bg\mathcal{C}$ (though perhaps not lying entirely inside A) has a common subsector with $\mathcal{C}$, so $\bar{w}bg = p \in P^o$, since P^o is the stabilizer of the face at infinity C_∞ of $\mathcal{C}$. Thus, $g = b^{-1}\bar{w}^{-1}p$, yielding the existence assertion of the theorem for G^o.

To prove that the indicated union is *disjoint* we must prove that the element $\bar{w}$ occurring above is uniquely determined as an element of the quotient

$$W_{\mathrm{sph}} = \mathcal{N}^o_{\mathrm{sph}} / \mathcal{N}^o_{\mathrm{trans}}$$

Consider two elements $b_1, b_2 \in B^o$ mapping subsectors $g\mathcal{C}_1, g\mathcal{C}_2$ (respectively) of $g\mathcal{C}$ to A. We may as well replace these two sectors by their intersection $g\mathcal{C}_o$. Now any minimal gallery from C to a chamber in $g\mathcal{C}_o$ lies in *every* apartment containing both C and $g\mathcal{C}_o$, by the combinatorial convexity of apartments (4.5). The automorphisms of X given by b_1, b_2 send non-stuttering galleries to non-stuttering galleries, agree pointwise on C, so on any apartment containing C and $g\mathcal{C}_o$ must be *equal*, by the Uniqueness Lemma (3.2).

That is, the actual *images* $b_1 g\mathcal{C}_o$, $b_2 g\mathcal{C}_o$ are the same. In particular, the *parallelism classes* of $b_i g\mathcal{C}_o$ are the same. Thus, the corresponding element $\bar{w}$

must be the same for any choice of $b \in B^o$ mapping a subsector of $g\mathcal{C}$ back to A. This proves the *uniqueness* part of the theorem for G^o.

Now we address G itself. We already know that $G = G^o \cdot T$ (5.5), so by invoking the theorem for G^o we have

$$G = G^o \cdot T = \bigcup_{\bar{w} \in W_{\mathrm{sph}}} P^o \bar{w} B^o \cdot T$$

$$= \bigcup_{\bar{w} \in W_{\mathrm{sph}}} P^o \bar{w} B^o \cdot (T^o \backslash T) = \bigcup_{\bar{w} \in W_{\mathrm{sph}}} P^o \bar{w} B^o \Omega$$

$$= \bigcup_{\bar{w} \in W_{\mathrm{sph}}, \sigma \in \Omega} P^o \bar{w} \sigma B^o$$

since T normalizes B^o (5.5).

For disjointness: if $P^o w_1 B^o t_1$ meets $P^o w_2 B^o t_2$ for $w_i \in W_{\mathrm{sph}}$ and $t_i \in T$, then surely $G^o t_1 = G^o t_2$. Then $T^o t_1 = T^o t_2$, so the images of t_1 and t_2 must be the same. This finishes the proof. ♣

Corollary: Let $K = K_x$ be the *good* 'maximal compact' subgroup

$$K = \bigsqcup_{\bar{w} \in W_x} B^o \bar{w} B^o \cdot \Omega$$

in G. (We assume throughout that S is finite, so an assumption that B^o is compact suffices to assure that this K is *literally* maximal compact. Then

$$G = P^o \cdot K$$

Proof: We have

$$G = \bigsqcup_{\bar{w}, \sigma} P^o \bar{w} \sigma B^o \subset \bigsqcup_{\bar{w}} P^o B^o \bar{w} \sigma B^o =$$

$$= P^o \cdot \bigsqcup_{\bar{w}, \sigma} B^o \bar{w} \sigma B^o = P^o K$$

as desired. ♣

17.7 Maximally strong transitivity

The point of this section is to see that when the Iwahori subgroup B is a *compact open* subgroup of G, then G^o acts strongly transitively on the *maximal* apartment system. Of course, this presumes that there is a *topology* on G so that this makes sense. A small amplification of the definition of *topological group* is appropriate.

A group G is a **topological group** if it has a topology in which the multiplication and inverse operations are *continuous*. That is, the maps $G \times G \to G$ by $g \times h \to gh$ and $G \to G$ by $g \to g^{-1}$ are both continuous. Most often a topological group is also required to be Hausdorff and locally compact, as well.

Of course, this definition has ramifications which are not obvious. A few simple observations are necessary for the sequel. For one, it follows that for every fixed $g \in G$ the maps $h \to gh$ and $h \to hg$ are continuous maps $G \to G$. Since these have the obvious inverses, they are *homeomorphisms*. As a consequence of this, for any open neighborhood U of the identity in G, gU and Ug are open neighborhoods of the point $g \in G$. Conversely, for any open neighborhood V of g, the sets $g^{-1}V$ and Vg^{-1} are open neighborhoods of the identity.

To relate this to the Iwahori subgroup B, *suppose that B is open and compact*. Because of the Bruhat-Tits decomposition (5.1), the assumption of open-ness implies that B is closed, since its complement is a union of sets Bg which are open, being continuous images (under the map $x \to xg$) of the open set B.

Proposition: Let Y be a subset of X which is contained in the union of finitely-many simplices in X. Suppose that Y contains at least one chamber. Then the pointwise fixer

$$G_Y = \{g \in G : gy = y \ \ \forall\, y \in Y\}$$

of Y is open and compact in G.

Proof: The hypothesis that Y contains a chamber C' entails that G_Y consists of label-preserving automorphisms, since every $g \in G_Y$ certainly preserves the labels on C', and by the Uniqueness Lemma (3.2) must preserve labels on any apartment containing that chamber. But by the building axioms (4.1) *every* chamber lies in some apartment containing C', so necessarily g preserves labels on the whole building.

If Y contains a point y in the closure of some chamber D, since $G_Y \subset G^o$, it must be that G_Y fixes the whole closure $\bar{D}$ pointwise. Thus, the pointwise fixer of Y is the same as the pointwise fixer of the smallest union of closed chambers containing Y.

Let $C_1, \ldots, C_n$ be the list of chambers whose closures contain Y. By hypothesis this list is finite. Invoking the transitivity of the label-preserving subgroup G^o of G on chambers, there is $h_i \in G^o$ so that $h_iC_i = C$, where C is the chamber whose pointwise fixer is B. Then

$$G_Y = \bigcap_i h_iBh_i^{-1}$$

This finite intersection of opens is open, and is certainly compact since each $h_iBh_i^{-1}$ is so. ♣

Theorem: With the hypothesis that B is compact and open in G, the group G^o of type-preserving maps in G acts strongly transitively on pairs $C' \subset A'$ for chambers C' and apartments A' in the *maximal* apartment system.

Proof: Let A' be an apartment in the maximal apartment system. We first reduce to the case that $C \subset A'$. Indeed, A' contains *some* chamber C', and by the mere transitivity of G^o on chambers there is $h \in G^o$ so that $hC' = C$. So now $C \subset hA'$, and if $hA' = gA$ for $g \in G^o$ then $A' = h^{-1}gA$. This is the desired reduction.

Now suppose that $C \subset A'$, where C is the distinguished chamber whose fixer is B. It suffices to find $b \in B$ so that $bA' = A$, where A is the distinguished apartment whose stabilizer in G^o is $\mathcal{N}$.

The simplicial complex A' is certainly the union of all its finite subcomplexes, so we can easily write it as a union $A' = \bigcup_i Y_i$ where

$$C \subset Y_1 \subset Y_2 \subset \ldots \subset Y_i \subset$$

and each Y_i is a *finite* chamber complex inside A'. (Note that this requires only that the Coxeter group W be *countable*, which is certainly assured by the uniform hypothesis that the generating set S be *finite*).

It was shown earlier (16.2) that the collection of *bounded* subsets of apartments does not depend upon the apartment system. Thus, each Y_i must lie in an apartment A_i in whatever apartment system $\mathcal{A}$ we start with, upon which G^o acts strongly transitively, by hypothesis.

Invoking the strong transitivity of G^o, there is $b_i \in B$ so that $b_i A_i = A$. For indices $i \leq j$ we have an isomorphism

$$b_i^{-1} b_j : A_j \to A_i$$

which gives the identity when restricted to Y_i. Thus, the sequence $b_1^{-1} b_i$ lies inside the compact set B, so has a convergent subsequence $b_1^{-1} b_{i_j}$ with limit β.

The obvious claim is that the element $b = b_1 \beta$ has the property that $bA' = A$. To prove this, let D be an arbitrary chamber in A. Choose i large enough so that the closure of D is contained in Y_i. Invoking the proposition, we can choose a small-enough neighborhood U of 1 in G so that $U \subset Y_i$. Choose j large enough so that $i_j \geq i$ and so that $b_1^{-1} b_{i_j} \in \beta U$. Then likewise

$$\beta \in b_1^{-1} b_{i_j} U \subset b_1^{-1} b_{i_j} Y_i$$

and

$$b = b_1 \beta \in b_{i_j} Y_i = b_i (b_i^{-1} b_{i_j}) Y_i$$

Since $i_j \geq i$, we have $b_i^{-1} b_{i_j} \in G_{Y_i}$, so

$$b \in b_i Y_i \cdot Y_i = b_i Y_i \subset A$$

by the defining property of b_i.

Then

$$bD \subset b_1 \beta Y_i \subset A$$

That is, the element $b \in B$ maps every chamber of A' to A. Thus, $bA' = A$. This proves that B is transitive on apartments in the *maximal* system containing C. This is the asserted strong transitivity. ♣

Corollary: If B is compact and open in G, then any apartment system $\mathcal{A}$ stable under the action of G is unavoidably the *maximal* apartment system. ♣

Remarks: The format of the previous theorem does not make clear what properties of the *building* might allow the Iwahori subgroup B to be compact and open, in some reasonable topology on G. However, in practice, often this is not the issue because the group G is *presented* with a topology arising from some other source.

17.8 Canonical translations

Keep notation as above. For this section we suppose that G preserves labels.

With fixed choice of apartment A and chamber C, let S denote the set of reflections through the facets of C. With fixed special vertex x of C, let S_x be the subset of S consisting of those reflections which fix x, and let W_x be the subgroup of W fixing x.

Attached to each $w \in W_x$ there is a *canonical translation*, usually denoted a_w, in the Levi component M of the minimal parabolic P, described as follows.

For $s \in S_x$, let F_s be the facet of C fixed by $s \in S$, and let η_s be the hyperplane which is the affine span of F_s. Thus, s is the reflection *through* η_s. Let F_o be the facet of C which does *not* contain x, and let η_o be the corresponding hyperplane. The chamber C is a simplex cut out by the hyperplanes η_s $(s \in S)$ and η_o (13.1), (13.6), (13.7).

Let W_{trans} be the subgroup of translations in W. The group W is the semi-direct product of W_x and W_{trans}. For each $w \in W_x$, write a semi-direct product decomposition

$$ws_o = a_w \cdot w'$$

with $a_w \in W_{\text{trans}}$ and $w' \in W_x$. *That is, a_w is the (uniquely-determined) translation so that*

$$(ws_o)x = a_w x$$

Thus, since $w \in S_x$, we also have

$$(ws_o w^{-1})x = a_w x$$

One notes that $ws_o w^{-1}$ is the reflection through the facet wF_o of the chamber wC. Thus, *a_w is a non-trivial translation in the direction orthogonal to the hyperplane wF_o.*

Proposition: The translations $\{a_w : w \in W_x\}$ generate a group Γ of finite index inside the group W_{trans} of all translations on A.

Proof: From prior discussion of the subgroup W_1 of *translations* of an affine Coxeter group W (12.4), to prove the finite-index assertion it suffices to show that the collection of all *directions* of the translations a_w span the space $|A|$.

Given any direction ν, consider a ray from the special vertex x in direction ν. An initial segment of ν must lie in (the closure of) some one of the chambers wC, $w \in W_x$, since the union of these is a neighborhood of x inside A. Thus, ν must intersect some facet wF_o for $w \in W_x$, where F_o is the facet of C opposite to x. Since the hyperplane $w\eta_o$ does not contain x, it must be that ν meets $w\eta_o$ at a non-zero angle. Thus, since the direction of a_w is orthogonal to $w\eta_o$, it cannot be that the direction of a_w and ν are orthogonal.

This proves that the collection of directions of all the translations a_w for $w \in W_x$ spans $|A|$. ♣

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18. Lattices, p-adic Numbers, Discrete Valuations

- p-adic numbers
- Discrete valuations
- Hensel's lemma
- Lattices
- Some topology
- Iwahori decomposition for $GL(n)$

As linear and geometric algebra formed the backdrop for the construction and application of *spherical* buildings, there is a corresponding bit of algebra which both motivates and is illuminated by the finer structure of affine buildings.

Fundamentally, the more delicate study of affine buildings is aimed at application to *p-adic groups*, the archetype for which is $GL(n, \mathbb{Q}_p)$. Thus, some exposition of the rudimentary properties of the p-adic *integers* $\mathbb{Z}_p$ and the p-adic *numbers* $\mathbb{Q}_p$ is appropriate. We need very little beyond the definitions.

On the other hand since many versions of this discussion take place in a broader context, we also introduce *discrete valuations* which generalize in a straightforward manner the p-adic numbers.

18.1 p-adic numbers

The definitions and simplest properties of p-adic numbers are all we need for later applieactions. Most of this material is really just an example of the *discrete valuation* scenario of the next subsection, but does deserve extra emphasis as the prototypical example.

The discussion of this section immediately generalizes to the more general case in which $\mathbb{Z}$ is replace by a principal ideal domain $\mathbf{o}$, the rational numbers $\mathbb{Q}$ are replace by the fraction field k of the principal ideal domain, and the prime number p is replaced by a generator π for a prime ideal in the principal ideal domain $\mathbf{o}$.

Let p be a prime number. The **p-adic valuation** is defined on the ordinary integers $\mathbb{Z}$ by

$$\text{ord}\, a\, p^n = \text{ord}_p\, a\, p^n = n$$

where a is an integer not divisible by p, and where n is a non-negative integer. Note that the fact that $\mathbb{Z}$ is a *unique factorization domain* entails that there is no ambiguity in the integer n appearing as exponent of p. By convention,

$$\text{ord}\, 0 = +\infty$$

Define the *p-adic norm* $| \, |_p$ on $\mathbb{Z}$ by

$$|x| = |x|_p = p^{-\text{ord } x}$$

and $|0|_p = 0$. The **p-adic metric** on $\mathbb{Z}$ is given by

$$d_{\text{p-adic}}(x, y) = |x - y|_p = |x - y|$$

The ring of **p-adic integers** $\mathbb{Z}_p$ is the completion of the ordinary integers $\mathbb{Z}$ with respect to the p-adic metric on $\mathbb{Z}$.

One definition of the field of **p-adic numbers** $\mathbb{Q}_p$ is as the field of fractions of this completion $\mathbb{Z}_p$. This is pointlessly indirect, however. It is better to define the p-adic *ord* function and *norm* and *metric* directly on $\mathbb{Q}$, and define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to this metric. To be sure that these two constructions yield the same thing one should check that *the ring operations in $\mathbb{Q}$ are continuous with respect to the topology from the p-adic metric.*

To directly define the p-adic valuation and norm on $\mathbb{Q}$: define ord_p on $\mathbb{Q}$ by

$$\text{ord } \frac{a}{b} p^n = \text{ord}_p(\frac{a}{b} \, p^n) = n$$

where $a, b \in \mathbb{Z}$ are both prime to p and b is non-zero. (And the *ord* of 0 is $+\infty$, again). Then the **p-adic norm** is

$$|x| = |x|_p = p^{-\text{ord } x}$$

Again, the fact that $\mathbb{Z}$ is a unique factorization domain implies that there is no ambiguity in the integer n appearing as exponent of p. The **p-adic metric** on $\mathbb{Q}$ is

$$d(x, y) = d_{\text{p-adic}}(x, y) = |x - y| = |x - y|_p$$

There is the visible multiplicative property

$$|xy|_p = |x|_p \; |y|_p$$

(which is what justifies calling this p-adic norm a *norm*). That this is so follows from the more elementary fact that if a prime p divides neither of two integers a, b, then p cannot divide the product ab.

That this is indeed a metric is easy to check: the symmetry is obvious, and an even *stronger* result, the **ultrametric inequality**, is obtained in place of the *triangle inequality*, as follows:

Proposition: For rational numbers x, y we have

$$|x + y|_p \le \max(|x|_p, |y|_p)$$

with *equality* holding unless $|x|_p = |y|_p$.

Proof: Write $x = p^m(a/b), y = p^n(c/d)$ with none of a, b, c, d divisible by p. Without loss of generality, by symmetry, we may suppose that $m \le n$. Then

$$x + y = p^m \, \frac{ad + p^{n-m}bc}{bd}$$

If $m < n$ then, since p divides neither a nor d, surely p does not divide the numerator. That is, if $m < n$ then equality holds in the statement of the proposition.

If $m = n$, then

$$x + y = p^m \, \frac{ad + bc}{bd}$$

and it may happen that p *does* divide the numerator, so that all we can conclude is that

$$\mathrm{ord}_p(x + y) \ge m$$

This gives the proposition. ♣

The effect of this completion is to annihilate information about any prime in $\mathbb{Z}$ other than p:

Proposition: Let x be any integer not divisible by a prime p. Then x is a *unit* in the p-adic integers $\mathbb{Z}_p$.

Remark: Further, the proof yields a 'formula' for the inverse of x.

Proof: Since the ideal $p\mathbb{Z}$ is maximal, the ideal $p\mathbb{Z} + x\mathbb{Z}$ must be the whole ring $\mathbb{Z}$. Thus, since $1 \in \mathbb{Z}$, there are $a, b \in \mathbb{Z}$ so that

$$ax + bp = 1$$

Evidently neither a nor b is divisible by p. Rearranging this, we have

$$ax = 1 - bp$$

and

$$\frac{a}{1 - bp} x = 1$$

So far this computation could take place inside the ordinary rational numbers $\mathbb{Q}$. But now we rewrite

$$\frac{a}{1 - bp} = a(1 + (bp) + (bp)^2 + (bp)^3 + \ldots)$$

with the assurance that the latter geometric series converges in $\mathbb{Q}_p$, since

$$|bp| = |b| \cdot |p| = |p| = \frac{1}{p} < 1$$

(since b is an integer prime to p). Then

$$x^{-1} = a(1 + (bp) + (bp)^2 + (bp)^3 + \ldots) \in \mathbb{Z}_p$$

since all the summands are ordinary integers. ♣

As a consequence of the last proposition, the p-adic integers $\mathbb{Z}_p$ contain all rational numbers of the form a/b with p not dividing the denominator b. Another paraphrase concerning this phenomenon is as follows:

Proposition: Let $x \in \mathbb{Z}_p$ and suppose that

$$|x - 1| < 1$$

Then x is a *unit* in $\mathbb{Z}_p$.

Proof: As in the last proof, we use the convergence of suitable geometric series. Supposing that $|x-1| < 1$, we have a convergent series

$$x^{-1} = (1-(1-x))^{-1} = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \ldots$$

Every summand is in $\mathbb{Z}_p$, so the convergent infinite sum yields an element of $\mathbb{Z}_p$. ♣

Corollary: Given a non-zero element x in $\mathbb{Z}_p$, for $y \in \mathbb{Z}_p$ sufficiently close to x, $y = \eta \cdot x$ for some unit η in $\mathbb{Z}_p$. Specifically, this holds if for y so that

$$|x - y| < |y|$$

And, in this situation, x and y necessarily generate the same ideal:

$$x\mathbb{Z}_p = y\mathbb{Z}_p$$

Proof: We have

$$y = x + (y-x) = x(1 + \frac{y-x}{x})$$

By the previous proposition, $1 + \frac{y-x}{x}$ is a unit, so by elementary ring theory x and y generate the same ideal. ♣

Proposition: The ring $\mathbb{Z}_p$ is a principal ideal domain *with only one non-zero prime ideal*, namely the ideal $\mathbf{m} = p\mathbb{Z}_p$ generated by p. Further, $\mathbf{m}$ is the set of elements $x \in \mathbb{Z}_p$ so that $|x|_p < 1$, and $\mathbb{Z}_p$ itself is the set of elements $x \in \mathbb{Q}_p$ so that $|x|_p \le 1$. The group of *units* $\mathbb{Z}_p^\times$ in $\mathbb{Z}_p$ is the set of elements x so that $|x|_p = 1$.

Proof: First, let's prove that the units are exactly the things in $\mathbb{Q}_p$ with norm 1. On one hand, if η is a unit, then η^{-1} lies in $\mathbb{Z}_p$, so $|\eta^{-1}| \le 1$ (as well as $|\eta| \le 1$). Then

$$1 = |1| = |\eta \cdot \eta^{-1}| = |\eta| \cdot |\eta^{-1}|$$

implies that $|\eta| = 1$.

On the other hand, suppose that $|\alpha| = 1$ for some $\alpha \in \mathbb{Q}_p$. Take $x, y \in \mathbb{Z}$ so that $|\alpha - \frac{x}{y}| < |\alpha|$. Then, by the proposition above, there is a unit η in $\mathbb{Z}_p$ so that $\alpha = \eta \cdot \frac{x}{y}$. Thus, since $|\alpha| = 1$, it must be that $|\frac{x}{y}| = 1$. Thus, the power of p dividing x must be identical to the power of p dividing y. We could have assumed that x, y are relatively prime, so then we conclude that *neither* x nor y is divisible by p. Thus, from above, they are both units in $\mathbb{Z}_p$. And then α itself must have been a unit.

Next, suppose that $|\alpha| \leq 1$. If $|\alpha| = 1$, then we have just seen that α is a *unit* in $\mathbb{Z}_p$. On the other hand, if $|\alpha| < 1$, then there is a power p^n of p (with $0 < n \in \mathbb{Z}$) so that $|\alpha/p^n| = 1$, so α/p^n is a unit, and certainly $\alpha \in p^n\mathbb{Z}_p$. This proves that $\mathbb{Z}_p$ (defined to be the completion of $\mathbb{Z}$ with respect to the p-adic metric) is exactly the set of elements in $\mathbb{Q}_p$ with norm less than or equal 1.

Now let I be a non-zero ideal in $\mathbb{Z}_p$. Let $x \in I$ be an element of I with maximal norm $|x|$ among all elements of I. This maximum really does occur, since the only possible values of the norm are

$$1, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots \to 0$$

(In particular, for any value of $|x_1|$, there are only finitely-many possible values above $|x_1|$ assumed on $\mathbb{Z}_p$). Then we claim that I is generated by this x. Indeed, for any other $y \in I$, $|y/x| = |y|/|x| \leq 1$, so by the previous argument $y/x \in \mathbb{Z}_p$ and $y \in x \cdot \mathbb{Z}_p$.

And, in particular, for any $x \in \mathbb{Q}_p$, there is some integer power p^n of p so that $x = \eta \cdot p^n$ with unit η in $\mathbb{Z}_p$. ♣

18.2 Discrete valuations

The object of this section is to run the ideas of the previous section in reverse, beginning with a 'discrete valuation' on a field, and from that constructing the 'discrete valuation ring', with properties analogous to $\mathbb{Z}_p$ above.

Just as in the p-adic case, there are two basic equivalent items: the *(discrete) valuation* and a *norm* (which engenders a *metric*). The norm is an exponentiated version of the valuation. The norm seldom has a canonical normalization, but this is usually not important.

A **discrete valuation** *ord* on a field k is an *integer-valued* function written $x \to \operatorname{ord} x$ on $k^\times$ so that

$$\operatorname{ord}(xy) = \operatorname{ord}(x) + \operatorname{ord}(y)$$

$$\operatorname{ord}(x+y) \geq \inf(\operatorname{ord}(x), \operatorname{ord}(y))$$

where we define $\operatorname{ord}(0) = +\infty$ compatibly. Very often the function *ord* is also called an '*ord-function*' or **ordinal**.

We assume that this *ord* function is *not* identically zero. Because of the multiplicative property, the collection of values of *ord* form a non-trivial additive subgroup of $\mathbb{Z}$. Thus, the collection of values is of the form $n\mathbb{Z}$ for some positive integer n. By replacing *ord* by $\frac{1}{n}$*ord*, we may assume without loss of generality that

$$\operatorname{ord} k^\times = \mathbb{Z}$$

For any real constant $c > 1$ there is a **norm** $r \to |r|$ on k associated to the valuation by

$$|r| = c^{-\mathrm{ord}\, r}$$

From the inequality $\mathrm{ord}(x + y) \geq \inf(\mathrm{ord}(x), \mathrm{ord}(y))$ we easily obtain the **ultrametric inequality**:

$$|x + y| \leq \max(|x|, |y|)$$

The field k is a **complete discretely-valued field** if it is complete as a metric space, with the obvious metric

$$d(x, y) = |x - y|$$

The associated **discrete valuation ring** is

$$\mathbf{o} = \{x \in k : |x| \leq 1\}$$

And define

$$\mathbf{m} = \{x \in k : |x| < 1\}$$

An element $\varpi \in \mathbf{o}$ so that

$$\mathrm{ord}\, \varpi = 1$$

is a **local parameter**.

Proposition: The valuation ring $\mathbf{o}$ really *is* a subring of k. The group of units $\mathbf{o}^\times$ in $\mathbf{o}$ is

$$\mathbf{o}^\times = \{x \in k : |x| = 1\}$$

The ring $\mathbf{o}$ is a principal ideal domain with unique non-zero prime ideal $\mathbf{m}$. And the *sharp* form of the ultrametric inequality holds: we have

$$|x + y| \leq \max(|x|, |y|)$$

with *equality* holding unless $|x| = |y|$.

Proof: Given x, y with $|x| \leq 1$ and $|y| \leq 1$, we must show that $|xy| \leq 1$ and $|x+y| \leq 1$. The multiplicative case is immediate, and the additive case follows because we have the *ultrametric* (rather than mere *triangle)* inequality. Thus, $\mathbf{o}$ really is a ring.

If $x \in \mathbf{o}$ has $|x| = 1$, then from

$$1 = |1| = |x \cdot x^{-1}| = |x| \cdot |x^{-1}|$$

we find that also $|x^{-1}| = 1$. Thus, x is a unit. The converse is clear.

Let I be a non-zero proper ideal. Let x be an element in I so that $\mathrm{ord}\, x$ is minimal among the values assumed by ord on I. (If the value 0 were assumed, then there would be units in I, contradiction). Then

$$\mathrm{ord}(x/\varpi^{\mathrm{ord}\, x}) = 0$$

so $x/\varpi^{\mathrm{ord}\, x}$ is a unit in $\mathbf{o}$. Thus,

$$I = \mathbf{o} \cdot \varpi^{\mathrm{ord}\, x}$$

This proves that $\mathbf{o}$ is a principal ideal domain.

Further, since every ideal is of the form $\mathbf{o}\varpi^n$ for some non-negative integer n, it is clear that $\mathbf{m} = \mathbf{o} \cdot \varpi$ is the only non-zero prime ideal.

To prove the sharp form of the ultrametric inequality, take $|y| < |x|$. Then

$$|x| = |(x+y) - y| \leq \max(|x+y|, |y|)$$

since $|-y| = |y|$. Since $|y| < |x|$, for this relation to hold it must be that

$$\max(|x+y|, |y|) = |x+y|$$

Putting this together, using the 'plain' ultrametric inequality, we have

$$|x| \leq |x+y| \leq \max(|x|, |y| = |x|$$

Then we have $|x| = |x+y|$ as asserted. ♣

18.3 Hensel's Lemma

For the present section we only need assume that k is a field with a non-negative real-valued **norm**

$$x \to |x|$$

which has the *multiplicative property*

$$|x \cdot y| = |x| \cdot |y|$$

and the *ultrametric property*

$$|x+y| \leq \max(|x|, |y|)$$

The associated metric is

$$d(x,y) = |x-y|$$

Such k is an **ultrametric field**. We assume that the norm $|\,|$ is *non-trivial*, meaning that $|1| = 1$, and also there is an element $\beta \in k$ with $|\beta| > 1$. We assume that k is *complete* with respect to this metric.

Proposition: There is the *sharp* ultrametric property: for $x, y \in k$

$$|x+y| = \max(|x|, |y|)$$

unless $|x| = |y|$.

Proof: This follows by the same proof as just above: take $|y| < |x|$. Then

$$|x| = |(x+y) - y| \leq \max(|x+y|, |y|)$$

which forces the maximum to be $|x+y|$, so

$$|x| \leq |x+y| \leq \max(|x|, |y| = |x|$$

and $|x| = |x+y|$ as asserted. ♣

The associated (not necessarily *discrete*) **valuation ring** is

$$\mathbf{o} = \{x \in k : |x| \leq 1\}$$

As in previous sections, it is the fact that we have the *ultrametric* inequality, rather than merely the triangle inequality, that makes $\mathbf{o}$ closed under addition.

In this context the analogue of *Newton's Method* works much more simply than would be the case if only the ordinary triangle inequality held for this norm.

Let f be a non-constant polynomial with coefficients in k. Write

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_o$$

with the a_i in k. The *derivative* $f'(x)$ can be defined purely algebraically, by the usual formula

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \ldots + 3a_3x^2 + 2a_2x + a_1$$

without taking any limits.

The usual Newton's method for iterative approximation of a root of a polynomial uses the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

starting from an initial approximation x_o, to determine a sequence of points x_1,x_2,$x_3, \ldots$ which presumably approach a root of f, that is, presumably

$$\lim_n \; f(x_n) =)$$

In the usual case of the real numbers, there is no simple hypothesis which will guarantee that this procedure yields a root. By contrast, in the ultrametric case things work out very nicely. As a simple but sufficient illustration, we have:

Theorem: Let k be a *complete ultrametric field* with valuation ring $\mathbf{o}$. Let $f(x)$ be a non-constant polynomial with coefficients in $\mathbf{o}$. Let $x_o \in \mathbf{o}$ so that

$$|f(x_o)| < 1$$

while

$$|f'(x)| = 1$$

holds. Then the sequence $x_1, x_2, x_3, \ldots$ defined recursively by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to a root of the equation $f(x) = 0$.

Proof: For any positive integer n

$$|n| = |1 + 1 + \ldots + 1| \leq |1| = 1 \quad \text{(with } n \text{ summands)}$$

Also, $|-1|^2 = |(-1)^2| = |1| = 1$, so $|-1| = 1$. Thus, (the image of) n in k lies in the valuation ring $\mathbf{o}$. For any positive integer ℓ and for any positive integer n

$$\frac{1}{\ell!}(\frac{d}{dx})^\ell x^n = (n)(n-1)(n-2)\ldots(n-(\ell-1))\, x^{n-\ell}$$

In particular, the coefficient is an *integer*. Therefore, if f is a polynomial with coefficients in $\mathbf{o}$, then all the polynomials $\frac{1}{i!}f^{(i)}$ also have coefficients in $\mathbf{o}$.

On purely algebraic grounds we have a *finite Taylor expansion*

$$f(x) = f(x_o) + f'(x_o)(x-x_o) + \frac{f^{(2)}(x_o)}{2!}(x-x)^2 + \ldots + \frac{f^{(m)}(x_o)}{m!}(x-x_o)^n$$

where m is the degree of f and $f^{(i)}$ indicates i^{th} derivative. (If the characteristic is *positive*, we must write the ratios $f^{(i)}/i!$ in a more sophisticated manner). The remarks just made assure that $f^{(i)}/i!$ has coefficients in $\mathbf{o}$.

Let x_o be as in the statement of the proposition. We will prove by induction that

- x_n lies in $\mathbf{o}$
- $|f'(x_n)| = 1$
- $|f(x_n)| \leq |f(x_o)|^{2^n}$

First, using a Taylor expansion for f', we have

$$f'(x_{n+1}) = f'(x_n) + f^{(2)}(x_n)(\frac{-f(x_n)}{f'(x_n)}) + \ldots + \frac{f^{(m-1)}(x_n)}{(m-1)!}(\frac{-f(x_n)}{f'(x_n)})^{m-1}$$

The first summand is a unit, while all the other summands have norm strictly less than 1. Thus, by the *sharp* ultrametric inequality, we conclude that $|f'(x_n)| = 1$.

Then, if x_n is in $\mathbf{o}$, since $f(x_n)$ is unavoidably in $\mathbf{o}$, it surely must be that x_{n+1} is again in $\mathbf{o}$.

By the Taylor expansion for f itself,

$$f(x_{n+1}) = f(x_n)+f'(x_n)(\frac{-f(x_n)}{f'(x_n)})+\frac{f^{(2)}(x_n)}{2!}(\frac{-f(x_n)}{f'(x_n)})^2+\ldots+\frac{f^{(n)}(x_n)}{n!}(\frac{-f(x_n)}{f'(x_n)})^n$$

which cancels to give

$$f(x_{n+1}) = \frac{f^{(2)}(x_n)}{2!}(\frac{-f(x_n)}{f'(x_n)})^2 + \ldots + \frac{f^{(m)}(x_n)}{m!}(\frac{-f(x_n)}{f'(x_n)})^m$$

Again using the fact that $f^{(i)}/i!$ has coefficients in $\mathbf{o}$, we have

$$|f(x_{n+1})| \leq |f(x_n)|^2$$

This proves the induction. ♣

18.4 Lattices

The notion of *lattice* which is relevant in this context is at some remove from more elementary and 'physical' concepts to which this word commonly refers, although the present version has its origins in the elementary ones.

Let k be the field of fractions of a discrete valuation ring $\mathbf{o}$. Let V be a finite-dimensional vectorspace over k. An $\mathbf{o}$-submodule Λ of V is an **o-lattice** if

- Λ is finitely-generated
- Λ contains a k-basis for V

For example, for any k-basis $e_1, \ldots, e_n$ for V, the $\mathbf{o}$-module

$$\Lambda = \mathbf{o}e_1 + \mathbf{o}e_2 + \ldots + \mathbf{o}e_n$$

is certainly an $\mathbf{o}$-lattice. In fact, every lattice is of this form:

Proposition: *Every* $\mathbf{o}$-lattice Λ in an n-dimensional k-vectorspace V is of the form

$$\Lambda = \mathbf{o}e_1 + \mathbf{o}e_2 + \ldots + \mathbf{o}e_n$$

for *some* k-basis $e_1, \ldots, e_n$ for V.

Proof: Let $e_1, \ldots, e_N$ be a minimal generating set for the $\mathbf{o}$-module Λ. (The existence of a minimal generating set follows from the finite generation). We will show that these elements are linearly independent over k. Let

$$0 = \alpha_1 e_1 + \ldots + \alpha_N e_N$$

be a relation, with $\alpha_i \in k$ not all zero. By renumbering if necessary, we may assume that $\operatorname{ord} \alpha_1$ is minimal among all the $\operatorname{ord} \alpha_i$. Then, dividing through by α_1, we have

$$m_1 = (-\alpha_2/\alpha_1) \cdot m_2 + \ldots + (-\alpha_N/\alpha_1) \cdot m_N$$

with all coefficients α_i/α_1 having non-negative ord, so lying in $\mathbf{o}$, by the previous section.

Since Λ is required to contain a k-basis for V, the elements of which would be expressible as $\mathbf{o}$-linear combinations of $e_1, \ldots, e_N$, it must be that the $e_1, \ldots, e_N$ themselves form a k-basis. ♣

18.5 Some topology

Let k be a field with a discrete valuation *ord* (with associated norm $|\,|$) on it. Let $\mathbf{o}$ be the valuation ring, with maximal ideal $\mathbf{m}$. We give k the topology from the metric

$$d(x, y) = |x - y|$$

associated to the valuation. And assume that k is *locally compact*. (This entails that k is *complete*, as well). Some features of this topology may be a little unexpected:

Proposition: The valuation ring $\mathbf{o}$, the group of units $\mathbf{o}^\times$, and the maximal ideal $\mathbf{m}$ are all simultaneously *open and closed* as subsets of k.

Proof: Let $|x| = c^{-\mathrm{ord}\, x}$ be the norm attached to the ord-function ord on k. Then

$$\mathbf{o} = \{x \in k : |x| < c\}$$

which shows that it is open, while at the same time its complement is

$$k - \mathbf{o} = \{x \in k : |x| > 1\}$$

which shows that $\mathbf{o}$ is closed as well. A nearly identical argument applies to the maximal ideal. Similarly,

$$\mathbf{o}^\times = \{x \in k : \frac{1}{c} < |x| < c\}$$

so $\mathbf{o}^\times$ is open, and its complement has a similar description, so $\mathbf{o}^\times$ is closed as well. ♣

We would also give the k-vectorspace k^n the product topology, which is readily seen to be equivalent to the *sup-norm topology* defined via

$$|(x_1, \ldots, x_n)| = \sup_i |x_i|$$

and the metric

$$d(x,y) = |x - y|$$

Let $GL(n,k)$ be the group of invertible n-by-n matrices with entries in k. We will specify a natural topology on $GL(n,k)$ so that the matrix multiplication of vectors

$$GL(n,k) \times k^n \to k^n$$

is continuous, so that matrix multiplication itself is continuous, and so that taking the inverse of a matrix is a continuous operation.

The most convenient description of the topology on $GL(n,k)$ is as follows. Let $M(n)$ be the n^2-dimensional k-vectorspace of n-by-n matrices with entries in k, with the product topology. Map

$$f : GL(n,k) \to M(n) \times M(n)$$

by

$$f(g) = (g, g^{-1})$$

and give $GL(n,k)$ the subspace topology from the product topology on $M(n) \times M(n)$. On the other hand, it will be convenient to know:

Proposition: For fixed $g \in GL(n,k)$, another element h in $GL(n,k)$ is close to g if and only if all the entries of h are close to those of g.

Proof: The 'only if' part follows from the definition of the topology on $GL(n,k)$). Note that this statement is *not* made *uniformly* in g, but only *pointwise* in g.

Define another *sup-norm*, now on matrices, by

$$|g| = \sup_{i,j} |g_{ij}|$$

where g_{ij} is the $(i,j)^{\text{th}}$ entry of g. The associated metric topology on the space $M(n)$ of n-by-n matrices is the same as the product topology on $M(n)$.

We first have a sub-multiplicativity property:

$$|gh| \le \sup_{i,\ell} |\sum_j g_{ij}h_{j\ell}| \le \sup_{i,\ell} \sup_j |g_{ij}h_{j\ell}| \le \sup_{i,j,i',j'} |g_{ij}| \cdot |h_{i'j'}| = |g| \cdot |h|$$

where use is made of the ultrametric inequality. This computation proves that matrix multiplication is continuous is this topology. A nearly identical computation proves that matrix multiplication of vectors are continuous in this topology.

What we must show is that, for fixed g, given $\epsilon > 0$ there is δ so that $|g-h| < \delta$ implies that $|g^{-1} - h^{-1}| < \epsilon$.

Let $h = g - \Delta$. Then

$$h^{-1} = (g-\Delta)^{-1} = [(1-\Delta g^{-1})g]^{-1}$$

$$= g^{-1}[1 + (\Delta g^{-1}) + (\Delta g^{-1})^2 + (\Delta g^{-1})^3 + \ldots]$$

if the latter series converges. This matrix-valued infinite series is entry-wise convergent in k if

$$|\Delta g^{-1}| < 1$$

In that case, also

$$|(\Delta g^{-1}) + (\Delta g^{-1})^2 + (\Delta g^{-1})^3 + \ldots| = |\Delta g^{-1}|$$

by the strict ultrametric inequality. Assuming $|\Delta g^{-1}| < 1$,

$$h^{-1} - g^{-1} = g^{-1}[(\Delta g^{-1}) + (\Delta g^{-1})^2 + (\Delta g^{-1})^3 + \ldots]$$

gives, by previous remarks and by the submultiplicativity,

$$|h^{-1} - g^{-1}| \le |g^{-1}| \cdot |\Delta g^{-1}| \le |g^{-1}| \cdot |\Delta| \cdot |g^{-1}|$$

This gives the desired continuity. ♣

The (standard) **Iwahori subgroup** B of $GL(n,k)$ is the set of matrices with

- Above-diagonal entries in $\mathbf{o}$
- Diagonal entries in $\mathbf{o}^\times$
- Below-diagonal entries in $\mathbf{m}$

Proposition: The Iwahori subgroup really *is* a subgroup, and for k locally compact, it is *compact and open* inside $GL(n,k)$.

Proof: The usual formula for the inverse of a matrix, as generally useless as it be, does suffice in this case to prove that the inverse of a matrix in B is again in B. More directly, the closure under matrix multiplication is easy to check. Note that the condition that the below-diagonal entries are in $\mathbf{m}$ is used in proving closure under matrix multiplication (and taking inverse).

Let g_{ij} be the $(i,j)^{\text{th}}$ entry of a matrix g. In $M(n)$, the set $\tilde{B}$ of matrices with diagonal entries units, above-diagonal entries in $\mathbf{o}$, and below-diagonal entries in $\mathbf{m}$, is a compact and open set, from the analogous observations on k itself, just above. Thus, the product of two copies of $\tilde{B}$ inside $M(n) \times M(n)$ is compact and open in the product topology. Thus, the intersection B of $\tilde{B} \times \tilde{B}$ with the copy $f(GL(n,k))$ of $GL(n,k)$ is compact and open in B. ♣

18.6 Iwahori decomposition for GL(n,k)

The decomposition result proven in this section for the Iwahori subgroup of $GL(n,k)$ has no analogue in more classical contexts.

As in the last section, B is the **Iwahori subgroup** of $GL(n,k)$ consisting of matrices whose diagonal entries are units in the valuation ring $\mathbf{o}$, whose above-diagonal entries are in $\mathbf{o}$, and whose below-diagonal entries are in the maximal ideal $\mathbf{m}$ of $\mathbf{o}$.

Let N be the subgroup of $GL(n,k)$ of upper-triangular matrices with 1's on the diagonal and 0's below the diagonal. Let N^{opp} be the subgroup of lower-triangular matrices with 1's on the diagonal and 0's above the diagonal. Let M be the subgroup of diagonal matrices in $GL(n,k)$. It bears emphasizing that these are sub*groups*, and not merely sub*sets*.

Theorem (Iwahori decomposition): Given an element b of the Iwahori subgroup B of $GL(n,k)$, there are uniquely-determined $u' \in N^{\text{opp}} \cap B$, $m \in M \cap B$, and $u \in N \cap B$ so that

$$b = u' \cdot m \cdot u$$

That is, B decomposes as

$$B = (N^{\text{opp}} \cap B) \cdot (M \cap B) \cdot (N \cap B)$$

and *uniquely* so.

Proof: We do an induction on the size n of the matrices involved. Specifically, we claim that for a given $b \in B$, we can find $u' \in N^{\text{opp}} \cap B$ and $u \in N \cap B$

so that $u' \cdot b \cdot u$ is of the form

$$u' \cdot b \cdot u = \begin{pmatrix} * & 0 & \ldots & 0 \\ 0 & * & \ldots & * \\ \vdots & & \ddots & \\ 0 & * & \ldots & * \end{pmatrix}$$

Indeed, if

$$b = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & * & \ldots & * \\ \vdots & & \ddots & \\ b_{n1} & * & \ldots & * \end{pmatrix}$$

then take

$$u' = \begin{pmatrix} 1 & 0 & \ldots & & 0 \\ -b_{11}^{-1}b_{21} & 1 & 0 & \ldots & 0 \\ -b_{11}^{-1}b_{31} & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -b_{11}b_{n1} & 0 & & & 1 \end{pmatrix}$$

and

$$u = \begin{pmatrix} 1 & -b_{11}^{-1}b_{12} & -b_{11}^{-1}b_{13} & \ldots & -b_{11}^{-1}b_{1n} \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & & \\ & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{pmatrix}$$

That is, u' differs from the identity matrix only in its left column, where the entries are designed to cancel the corresponding entries of b upon left multiplication by u'. Likewise, u differs from the identity matrix only in its first row, where the entries are designed to cancel the corresponding entries of b upon right multiplication by u. All the entries of u and u' are in $\mathbf{o}$ since b_{11} is a unit in $\mathbf{o}$. It is immediate that $u' \cdot b \cdot u$ has the desired form.

The induction proceeds by viewing the lower right $(n-1)$-by-$(n-1)$ block of an n-by-n matrix as a matrix in its own right, recalling that matrix multiplication behaves well with respect to blocks:

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0 & DD' \end{pmatrix}$$

where the 0's denote appropriately-shaped blocks of zeros, A and A' are square matrices of the same size, and D and D' are square matrices of the same size. ♣

Remarks: Note that neither completeness nor local compactness played a role in this argument.

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19. Affine Constructions for SL(n)

- Construction of the affine building for SL(V)
- Verification of the building axioms
- Action of SL(V) on the affine building
- The Iwahori subgroup 'B'
- The maximal apartment system

Here we give a construction which is the simplest example of an *affine* building and BN-pair. The material objects involved in the construction were appreciated long before their roles in an affine building construction were understood at all.

The affine building constructed here is attached to a vectorspace V over the fraction field k of a discrete valuation ring $\mathbf{o}$. For the finer results it will be assumed that the discrete valuation ring is *complete* (with respect to the metric attached to the valuation), and probably *locally compact*. These hypotheses certainly hold in the p-adic case, which is the case of fundamental practical importance.

The corresponding group which will act nicely on the building is $G = SL(V)$, the group of k-linear automorphisms of V which have determinant 1.

We will see that the apartments are Coxeter complexes attached to the Coxeter system (W, S) of type $\tilde{A}_{n-1}$ described earlier (2.2). The fact that this truly is *affine*, verified in terms of the Coxeter data criterion (13.6), was done in (13.8), so all we need to do here is to check that the Coxeter data is as claimed.

This standard notation does suggest, among other things, that omission of the generator s_o from the Coxeter system leaves us with a group of type A_{n-1}, that is, a symmetric group on n things. From looking at the Coxeter data, this is indeed the case. And thus the spherical building at infinity is of type A_{n-1}, which is to say that the Coxeter complexes which are the apartments are of that type.

19.1 Construction of the affine building for SL(V)

Here we construct the simplest example of a *thick affine building*. It happens that the apartment system we describe here is the *maximal* one if the discrete valuation ring involved is *complete*.

As in every other case, the procedure is that we describe an *incidence geometry* from which we obtain a *flag complex* which we verify is a *thick building* by checking the axioms. Once we identify the Coxeter data as being

$\tilde{A}_{n-1}$, computations already done (13.8) assure that the building is indeed *affine*.

Let $\mathbf{o}$ be a discrete valuation ring with fraction field k and unique non-zero prime ideal $\mathbf{m}$. Let $\kappa = \mathbf{o}/\mathbf{m}$ be the residue field. Let ϖ be a *local parameter*, that is, a generator for $\mathbf{m}$.

Let V be an n-dimensional vectorspace over k. Take $G = SL(V)$, the k-linear automorphisms of V which act trivially on the n^{th} exterior power of V (that is, which have determinant one, as matrices).

A **homothety** $f : V \to V$ is a k-linear map $v \to \alpha v$ for some $\alpha \in k^\times$. That is, a homothety is a non-zero scalar multiplication. Two ($\mathbf{o}$-)lattices Λ, Λ' are **homothetic** if there is a homothety $v \to \alpha v$ so that $\alpha\Lambda = \Lambda'$. Being homothetic is an *equivalence relation*; we write $[\Lambda]$ for the homothety (equivalence) class of a lattice Λ.

Take the set of *vertices* Ξ for our incidence geometry to be the set of **homothety classes of lattices** in V. We have an *incidence relation* $\sim$ on Ξ defined as follows: write $\xi \sim \eta$ for $\xi, \eta \in \Xi$ if there are $x \in \xi$ and $y \in \eta$ so that $y \subset x$ and on the quotient $\mathbf{o}$-module x/y we have $\mathbf{m} \cdot x/y = 0$. (Thus, the quotient has a natural structure of vectorspace over the residue field κ.)

Let's check that this relation really is symmetric: with representatives x, y as just above, let $y' = \mathbf{m}y$. Then

$$\mathbf{m}x \subset y' \subset x$$

where $\mathbf{m}x \subset y'$ follows from $x \subset y$ by multiplying by $\mathbf{m}$.

It is important to realize that if two homothety classes $[L], [M]$ are incident then *any* two representatives L, M have the property that *either* $L \subset M$ or $L \supset M$. To see this, first take representatives L, M so that $\mathbf{m}M \subset L \subset M$. Let m, n be arbitrary integers. Certainly if $m \geq n$ then

$$\mathbf{m}^m L \subset \mathbf{m}^m M \subset \mathbf{m}^n M$$

On the other hand, if $m < n$ then $n - 1 \geq m$ and

$$\mathbf{m}^n M = \mathbf{m}^{n-1}(\mathbf{m}M) \subset \mathbf{m}^{n-1} L \subset \mathbf{m}^m L$$

Thus, one or the other of the two inclusions must hold. Things are not this simple for arbitrary homothety classes.

As defined earlier, the associated *flag complex* X is the simplicial complex with vertices Ξ and simplices which are *mutually incident* subsets of Ξ, that is, subsets σ of Ξ so that, for all $x, y \in \sigma$, $x \sim y$.

In the present context, a **frame** is an unordered set $\lambda_1, \ldots, \lambda_n$ of lines (one-dimensional k-subspaces) in V so that

$$\lambda_1 + \ldots + \lambda_n = V$$

We take a set $\mathcal{A}$ of subcomplexes indexed by frames $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ in V as follows: the *associated apartment* $A = A_{\mathcal{F}} \in \mathcal{A}$ consists of all simplices σ

with vertices $[\Lambda]$ which are homothety classes of lattices with representative Λ expressible as

$$\Lambda = L_1 + \ldots + L_n$$

where L_i is a *lattice* in the line (one-dimensional vector space) λ_i.

It will be very convenient to know that the *maximal simplices* in the simplicial complex X are in bijection with ascending chains of lattices

$$\ldots \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_{n-1} \subset \Lambda_n \subset \ldots$$

(indexed by integers) where there is the *periodicity*

$$\Lambda_{i+n} = \mathbf{m}\Lambda_i$$

for all indices i, and where for all i the quotient Λ_{i+1}/Λ_i is annihilated by $\mathbf{m}$ and is a *one-dimensional* κ-vectorspace. This corresponds to the maximal mutually incident set

$$[\Lambda_0],\ [\Lambda_1], \ldots ,\ [\Lambda_{n-1}]$$

of homothety classes of lattices.

Indeed, we claim that if $[x_1], \ldots , [x_n]$ are mutually incident then, re-ordering (renumbering) if necessarys, there are representatives $x_1, \ldots , x_n$ so that

$$\ldots \subset \mathbf{m}x_n \subset x_1 \subset x_2 \subset x_3 \subset \ldots \subset x_n \subset \mathbf{m}^{-1}x_1 \subset \ldots$$

This is proven by induction on n. Suppose that we already have

$$\ldots \subset \mathbf{m}x_n \subset x_1 \subset x_2 \subset x_3 \subset \ldots \subset x_\ell \subset \mathbf{m}^{-1}x_1 \subset \ldots$$

and are given another homothety class $[y]$ incident to all the $[x_i]$. Choose a representative y for this class so that

$$\mathbf{m}y \subset x_\ell \subset y$$

invoking the fact that $y \sim x_\ell$.

If it should happen that $\mathbf{m}y \subset x_1$, then we are done, since

$$\ldots \subset \mathbf{m}y \subset x_1 \subset \ldots \subset x_\ell \subset y \subset \ldots$$

is the desired configuration.

Otherwise, there is a minimal index i so that $\mathbf{m}y \subset x_i$. And $i \leq \ell$ since $\mathbf{m}y \subset x_\ell$. Since $[x_{i-1}]$ and $[y]$ are incident, it follows that $x_{i-1} \subset \mathbf{m}y$. But then we replace the representative y by the better representative $\mathbf{m}y$ and the configuration

$$\ldots \subset \mathbf{m}x_\ell \subset x_1 \subset \ldots \subset x_{i-1} \subset \mathbf{m}y \subset x_i \subset \ldots \subset x_\ell \subset \mathbf{m}^{-1}x_1 \subset \ldots$$

is as desired.

It is easy to go in the other direction, from such an infinite periodic flag to a maximal mutually incident collection of homothety classes. Thus, we have proven that maximal families of mutually incident homothety classes are essentially the same things as infinite periodic flags.

19.2 Verification of the building axioms

Keep all the notation of the previous section.

Given a maximal simplex C, the i^{th} *facet* F_i is obtained by omitting Λ_i and also omitting all $\Lambda_{i+\ell n}$ for $\ell \in \mathbb{Z}$. Any other maximal simplex with facet F_i is obtained by choice of lattices $\Lambda'_{i+\ell n}$ meeting three conditions:

$$\Lambda'_{i+\ell n} = \mathbf{m}^{\ell}\Lambda'_i$$

and

$$\Lambda_{i-1+\ell n} \subset \Lambda'_{i+\ell n} \subset \Lambda_{i+1+\ell n}$$

and where Λ'_i/Λ_{i-1} is a one-dimensional subspace of the two-dimensional κ vectorspace $\Lambda_{i+1}/\Lambda_{i-1}$

Let A be the apartment corresponding to the frame $\lambda_1, \ldots, \lambda_n$. Let C be a maximal simplex in A corresponding to a periodic flag $\ldots \subset \Lambda_i \subset \ldots$ of lattices, as above. For a fixed index i, let F_i be the facet of C corresponding to omission of the lattices $\Lambda_{i+\ell n}$. As just noted, the question of finding all other maximal simplices in A with facet F_i is just the question of finding other families $\Lambda'_{i+\ell n}$ with which to replace $\Lambda_{i+\ell n}$. Since $\Lambda_{i+1}/\Lambda_{i-1}$ is two-dimensional over κ, there are exactly two indices j_1, j_2 so that $\lambda_{j_1} \cap \Lambda_{i+1}$ and $\lambda_{j_2} \cap \Lambda_{i+1}$ map surjectively to $\Lambda_{i+1}/\Lambda_{i-1}$. Then between the two lattices

$$\Lambda_{i-1} + (\lambda_{j_1} \cap \Lambda_{i+1}) \qquad \Lambda_{i-1} + (\lambda_{j_2} \cap \Lambda_{i+1})$$

one must be Λ_i, and the other is the unique *other* candidate Λ'_i to replace Λ_i. Thus, if the apartment A is a *chamber complex* then it is *thin*.

Likewise, it is easy to see the *thickness* of the building: in the building, the choices for replacement Λ'_i are in bijection with one-dimensional κ-subspaces of the two-dimensional κ-vectorspace $\Lambda_{i+1}/\Lambda_{i-1}$, other than Λ_i/Λ_{i-1}. If κ is infinite we are surely done. If κ is finite with cardinality q, then there are

$$(q^2-1)/(q-1) = q+1 > 2$$

maximal simplices with facet F_i. This proves the thickness of the whole building (granting that it is a chamber complex).

Now we show that the apartment A *is* a chamber complex by showing that there is a gallery from C to any other maximal simplex. To see this, we consider the effect of 'reflecting' across the facets of maximal simplices.

Choose lattices M_i in λ_i so that

$$\begin{array}{ccl}
\Lambda_o & = & M_1 + \ldots + M_n \\
\Lambda_1 & = & \mathbf{m}^{-1}M_1 + M_2 + \ldots + M_n \\
\ldots & & \\
\Lambda_i & = & \mathbf{m}^{-1}(M_1 + \ldots + M_i) + M_{i+1} + \ldots + M_n \\
 & \ldots & \\
\Lambda_{n-1} & = & \mathbf{m}^{-1}(M_1 + \ldots + M_{n-1}) + M_n
\end{array}$$

where the lattices Λ_i are those appearing in the flag describing the maximal simplex C. Note that the set of vertices of simplices in A consists of homothety classes of lattices which can be expressed as

$$\mathbf{m}^{m_1} M_1 + \ldots + \mathbf{m}^{m_n} M_n$$

for some n-tuple of integers $(m_1, \ldots, m_n)$.

As above, let F_i be the facet of C obtained by omitting $\Lambda_{i+\ell n}$. As in the discussion of thin-ness and thick-ness above, for $1 \leq i < n$, the unique *other* maximal simplex with facet F_i is obtained by replacing

$$\Lambda_i = \mathbf{m}^{-1}(M_1 + \ldots + M_i) + M_{i+1} + \ldots + M_n$$

by

$$\Lambda_i' = \mathbf{m}^{-1}(M_1 + \ldots + M_{i-1}) + M_i + \mathbf{m}^{-1} M_{i+1} + \ldots + M_n$$

That is, *reflecting through F_i has the effect of reversing the roles of M_i* and M_{i+1} (for $1 \leq i < n$).

If $i = 0$, then the analogous conclusion is that reflection through $F_i = F_o$ causes $M_1, \ldots, M_n$ to be replaced by $M_1', \ldots, M_n'$ with

$$M_1' = \mathbf{m} M_n$$

$$M_n' = \mathbf{m}^{-1} M_1$$

$$M_i' = M_i' \text{ for } \quad 1 < i < n$$

As noted in our prior discussion of the spherical building for $GL(n)$ over a field, it is elementary that the collection of interchanges of i and $i+1$ generate the group of permutations of $1, 2, 3, \ldots, n$. Thus, by such interchanges, we can go from the ordering

$$M_1, \ldots, M_n$$

to the ordering

$$M_1, \ldots, \hat{M}_i, \ldots, M_n, M_i$$

that is, move a chosen M_i to the right end of this ordering. The reflection through F_o turns this into the ordering

$$\mathbf{m} M_i, M_2, \ldots, \hat{M}_i, \ldots, M_n, \mathbf{m}^{-1} M_1$$

The adjacent interchanges can be used to permute this back to

$$\mathbf{m}^{-1} M_1, M_2, \ldots, M_{i-1}, \mathbf{m} M_i, M_{i+1}, \ldots, M_n$$

By a composition of such reflections, we can replace any M_i (with $i > 1$) by $\mathbf{m}^{m_i} M_i$, at the cost of replacing M_1 by $\mathbf{m}^{-m_i} M_1$. We can then arbitrarily permute the resulting lattices, by use of the adjacent interchanges. *Up to homothety*, such manipulations can give an arbitrary flag of lattices. *Thus, the apartments are (thin) chamber complexes.*

Next, we will prove that any two maximal simplices lie inside a common apartment. (In light of the previous paragraph, this will also prove that the whole building really is a chamber complex). Let C, D be two chambers corresponding to *periodic* flags

$$\mathcal{F} = (\ldots \subset \Lambda_i \subset \ldots)$$

$$\mathcal{F}' = (\ldots \subset \Lambda'_i \subset \ldots)$$

where the successive quotients are one-dimensional κ-vectorspaces, as above.

The filtration of V given by $\mathcal{F}'$ gives a filtration of each quotient Λ_{i+1}/Λ_i attached to $\mathcal{F}$, permitting application of a Jordan-Holder-type argument nearly identical to the argument used for the *spherical* construction for $GL(n)$: that is For each i, we have a *filtration* of Λ_i/Λ_{i-1} given by the Λ'_j:

$$\ldots \subset \frac{(\Lambda_i \cap \Lambda'_j) + \Lambda_{i-1}}{\Lambda_{i-1}} \subset \ldots$$

For all indices i, j we have

$$\frac{\Lambda_i}{\Lambda_{i-1}} \supsetarrow \frac{(\Lambda_i \cap \Lambda'_j) + \Lambda_{i-1}}{\Lambda_{i-1}} \xrightarrow{\text{onto}} \frac{(\Lambda_i \cap \Lambda'_j) + \Lambda_{i-1}}{\Lambda_{i-1} + (\Lambda_i \cap \Lambda'_{j-1})} \approx \frac{\Lambda_i \cap \Lambda'_j}{(\Lambda_{i-1} \cap \Lambda'_j) + (\Lambda_i \cap \Lambda'_{j-1})}$$

The space Λ_i/Λ_{i-1} is one-dimensional over κ, so for given i there is a *first* index j for which the quotient

$$\frac{(\Lambda_i \cap \Lambda'_j) + \Lambda_{i-1}}{\Lambda_{i-1}}$$

is one-dimensional over κ. With this j, we *claim* that

$$\Lambda_i \cap \Lambda'_{j-1} \subset \Lambda_{i-1}$$

If not, then

$$\Lambda_i = \Lambda_{i-1} + (\Lambda_i \cap \Lambda'_{j-1})$$

since the dimension of Λ_i/Λ_{i-1} is one. But by its definition, j is the smallest among indices ℓ so that

$$\Lambda_i = \Lambda_{i-1} + (\Lambda_i \cap \Lambda'_\ell)$$

Thus, the claim is proven.

Thus, given i, there is exactly one index j for which

$$\frac{\Lambda_i \cap \Lambda'_j}{(\Lambda_{i-1} \cap \Lambda'_j) + (\Lambda_i \cap \Lambda'_{j-1})}$$

is one-dimensional. The latter expression is symmetrical in i and j, so there is a permutation π of the set of integers so that this expression is one-dimensional only if $j = \pi(i)$ and otherwise is 0.

For some maximal index i_o, for all $i \leq i_o$ we have $\Lambda'_i \subset \Lambda_o$, since for all indices m we have the *periodicity* $\Lambda'_{m-n} = \mathbf{m}\Lambda'_m$. The flag $\mathcal{F}$ has the same *periodicity* property $\Lambda_{m-n} = \mathbf{m}\Lambda_m$. Requiring preservation of this periodicity, the permutation π is completely determined by $\pi(0), \pi(1), \ldots, \pi(n-1)$, which must lie among $i_o, i_o + 1, \ldots, i_o + n - 1$.

At this point it is possible to give a frame specifying an apartment containing both chambers, as follows: For $i = 0, 1, \ldots, n-1$ let M_i be a free rank-one $\mathbf{o}$-module in $\Lambda_i \cap \Lambda'_{\pi(i)}$ which maps *onto* the κ-one-dimensional quotients. Then put

$$\lambda_i = kM_i$$

The unordered set of lines $\lambda_1, \ldots, \lambda_n$ is the desired frame. *Thus, we have verified one building axiom, that any two chambers lie in a common apartment.*

Also, since we have proven that the (alleged) apartments really are chamber complexes, we have proven that the whole complex really is a chamber complex, that is, any two maximal simplices are connected by a gallery.

Last, we verify the other building axiom: given a simplex x and a chamber C both lying in two apartments A, B, show that there is an isomorphism $B \to A$ fixing both x and C pointwise. We will in fact give the map by giving a bijection between rank-one $\mathbf{o}$-modules describing the respective frames, possibly changing by homothety. This surely would give a face-relation-preserving bijection between the simplices. And, as in all other examples, it turns out to be simpler to prove the apparently stronger assertion that, given two apartments A, B containing a chamber C, there is an isomorphism $f : B \to A$ fixing $A \cap B$ pointwise.

Let $\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$ and $\mathcal{G} = \{\mu_1, \ldots, \mu_n\}$ be unordered lists of lines specifying the apartments A, B, respectively. Without loss of generality, we can renumber so that the chamber C corresponds to orderings

$$(M_1, \ldots, M_n) \text{ and } (N_1, \ldots, N_n)$$

where M_i, N_i are rank-one $\mathbf{o}$-modules inside λ_i, μ_i, respectively. That is, the lattice homothety classes occurring as vertices of C are

$$[\Lambda_i] = [\mathbf{m}^{-1}(M_1 + \ldots + M_i) + M_{i+1} + \ldots + M_n] =$$
$$= [\mathbf{m}^{-1}(N_1 + \ldots + N_i) + N_{i+1} + \ldots + N_n]$$

Since these homothety classes must be the same for all indices i, it is easy to see that (changing by a homothety) we can suppose that

$$\Lambda_i = \mathbf{m}^{-1}(M_1 + \ldots + M_i) + M_{i+1} + \ldots + M_n$$
$$= \mathbf{m}^{-1}(N_1 + \ldots + N_i) + N_{i+1} + \ldots + N_n$$

Consider the map

$$f : B \to A$$

given on lattices by

$$\mathbf{m}^{m_1} M_1 + \ldots + \mathbf{m}^{m_n} M_n \to \mathbf{m}^{m_1} N_1 + \ldots + \mathbf{m}^{m_n} N_n$$

for any integers $m_1, \ldots, m_n$. By its nature, this map respects homothety classes, as required.

To show that f is the identity on $A \cap B$ it suffices to show that it is the identity on all 0-simplices in the intersection. If a 0-simplex $[x]$ lies in $A \cap B$

then $[x]$ is a homothety class of lattices which has a representative x which can be written as

$$x = \mathbf{m}^{m_1} M_1 + \ldots + \mathbf{m}^{m_n} M_n$$

and also as

$$x = \mathbf{m}^{\mu_1} N_1 + \ldots + \mathbf{m}^{\mu_n} N_n$$

We will show that $m_i = \mu_i$ for all indices i, thereby certainly assuring that all of $A \cap B$ is fixed pointwise by f.

Let i_o be the largest index so that $m_{i_o} = \min\{m_i\}$, and let j_o be the largest index so that $\mu_{j_o} = \min\{\mu_j\}$. On onte hand, if $m_{i_o} < \mu_{j_o}$, then

$$(\mathbf{m}^{m_1} M_1 + \ldots + \mathbf{m}^{m_n} M_n)/\mathbf{m}^{\mu_{j_o}} \Lambda_o$$

requires at least one generator as $\mathbf{o}$-module, but, on the other hand,

$$(\mathbf{m}^{\mu_1} N_1 + \ldots + \mathbf{m}^{\mu_n} N_n)/\mathbf{m}^{\mu_{j_o}} \Lambda_o = 0$$

so needs zero generators as $\mathbf{o}$-module, contradicting the hypothesis that these two modules are simply different expressions for $x/\mathbf{m}^{\mu_{j_o}} \Lambda_o$. Thus, by symmetry, it must be that $m_{i_o} = \mu_{j_o}$.

Further, to show that $i_o = j_o$, suppose that $i_o < j_o$, and consider

$$x/\mathbf{m}^{m_{i_o}+1} \Lambda_{i_o}$$

Viewed in the M_i coordinates, this quotient module is 0, that is, has zero generators. Viewed in the N_j coordinates, this quotient needs at least one generator, contradiction. Thus, $i_o = j_o$.

This is the beginning of an induction which proves that $m_i = \mu_i$ for all indices i. That is, f is the identity map on $A \cap B$. *This completes the proof that we have constructed a building.*

Review of this discussion makes clear that the Coxeter data is as indicated at the beginning of this section: Let s_i be the reflection through the i^{th} facet F_i, with $i = 0, 1, 2, \ldots, n-1$. Designate a chamber in an apartment by an ordered set $(M_1, \ldots, M_n)$ of free rank-one $\mathbf{o}$-modules in V so that the sum spans V over k.

If $n = 2$, then in the notation above,

$$(M_1, M_2) \xrightarrow{s_o} (\mathbf{m}M_2, \mathbf{m}^{-1}M_1) \xrightarrow{s_1} (\mathbf{m}^{-1}M_1, \mathbf{m}M_2)$$

by our earlier computations. Thus, $s_1 s_o$ is of infinite order.

If $n > 2$ and $i - j$ is not ± 1 *modulo* n, then s_i and s_j certainly commute. If $1 \le i < n-1$ and $j = i+1$, then

$$(\ldots, M_i, M_{i+1}, M_{i+2}, \ldots) \xrightarrow{s_i} (\ldots, M_{i+1}, M_i, M_{i+2}, \ldots)$$

$$\xrightarrow{s_{i+1}} (\ldots, M_{i+1}, M_{i+2}, M_i, \ldots)$$

by earlier computations. Thus, $s_{i+1} s_i$ is of order 3, as asserted at the beginning of this section.

Thus, not only have we verified that this construction gives a thick building, but we also have determined the Coxeter data so as to confirm (in light of earlier computations for our seven families of Coxeter systems) that this system is *affine*.

19.3 The action of SL(V) on the affine building

Now we check that the natural group action of $SL(V)$ on the (affine) building just constructed is *type-preserving* and *strongly transitive*. Thus, we obtain an *affine BN-pair* which is discussed in the next subsection.

Remarks: In fact, although $GL(V)$ will not preserve labels, the subgroup G^+ of $GL(V)$ consisting of automorphisms whose determinant has *ord* divisible by n, the dimension, *does* preserve the labeling. (As usual the *ord* function is defined by $\mathrm{ord}\,\alpha = n$ where $\alpha\mathbf{o} = \mathbf{m}^n$.

As in the earlier discussions of examples of *spherical* buildings, as soon as we have a congenial notation the proofs become easy.

As in the case of the spherical buildings earlier, it is convenient to use a concrete labeling, as follows. Fix one vertex $[\Lambda_o]$ of C, where Λ_o is a lattice and $[\Lambda_o]$ is its homothety class. Given any other homothety class $[\Lambda]$, we may choose a representative Λ so that $\Lambda_o \subset \Lambda$. The quotient Λ/Λ_o is a finitely-generated torsion $\mathbf{o}$-module isomorphic to

$$\mathbf{o}/\mathbf{m}^{d_1} \oplus \ldots \oplus \mathbf{o}/\mathbf{m}^{d_n}$$

with some non-negative integers $d_1 \leq \ldots \leq d_n$. Define a labeling by

$$\nu([\Lambda]) = \sum_i d_i \mod n$$

This function ν certainly gives a labeling of vertices, and thereby a labeling of simplices. Now the action of elements of $G = SL(V)$ actually preserves not only ν, but in fact preserves $\sum_i d_i$ without reducing modulo n. *Thus, the action of $G = SL(V)$ preserves this labeling.*

Remarks: At this point it is also clear that the funny subgroup G^+ of $GL(V)$ consisting of those automorphisms with determinant having *ord* divisible by n is the label-preserving subgroup of $GL(V)$. Proving the strong transitivity for $G = SL(V)$ certainly suffices to prove it for this G^+.

The ordinary transitivity of the group on apartments is straightforward: apartments are designated by unordered n-tuples (frames) $\mathcal{F} = \{\lambda, \ldots, \lambda_n\}$ of lines in V so that $V = \sum \lambda_i$. Certainly $SL(V)$ is transitive on these, as was already used in the discussion of the spherical examples.

We must check that the stabilizer of an apartment acts transitively on the set of chambers within that apartment.

The stabilizer $\mathcal{N}$ of an apartment A specified by a frame

$$\mathcal{F} = \{\lambda_1, \ldots, \lambda_n\}$$

is the group of linear maps which stabilize the set of lines λ_i making up the frame. Thus, the stabilizer $\mathcal{N}$ certainly includes linear maps to give arbitrary *permutations* of the lines $\lambda_1, \ldots, \lambda_n$. Further, for any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of elements of $k^\times$ so that $\Pi_i \, \alpha_i = 1$, we have the map $g = g_\alpha \in SL(V)$ given by multiplication by α_i on λ_i.

A chamber in an apartment can be described by a (periodic) flag

$$\ldots \subset \Lambda_o \subset \Lambda_1 \subset \ldots \subset \Lambda_{n-1} \subset \ldots$$

of lattices Λ_i in V, where, possibly renumbering the λ_i,

$$\Lambda_o = M_1 + \ldots + M_n$$

and generally

$$\Lambda_i = \mathbf{m}^{-1}(M_1 + \ldots + M_{i-1}) + M_i + \ldots + M_n$$

and there is the periodicity

$$\Lambda_{i-n} = \mathbf{m} \, \Lambda_i$$

Keep in mind that we must allow ambiguity by *homotheties*, and that we can let Λ_o have whatever *type* we choose.

The action of $g = g_\alpha$ in this notation is

$$g_\alpha(M_1, \ldots, M_n) = (\alpha_1 M_1, \ldots, \alpha_n M_n)$$

And arbitrary permutations of the lines can be achieved by determinant-one matrices. Thus, with the type restriction and allowing for homotheties, we have the desired strong transitivity.

19.4 The Iwahori subgroup 'B'

Now we want to identify the Iwahori subgroup 'B', defined as being the stabilizer of a chamber in the affine building.

We will see that, with suitable choices and coordinates, the Iwahori subgroup B is the collection of matrices in $SL(n, \mathbf{o})$ which modulo $\mathbf{m}$ are *upper-triangular*. That is, if $g_{i,j}$ is the $(i,j)^{\text{th}}$ entry of an element $g \in SL(n, \mathbf{o})$, then we require that

$$g_{i,j} \in \mathbf{o} \quad \text{for} \quad i \leq j$$
$$g_{i,j} \in \mathbf{m} \quad \text{for} \quad i > j$$

(Of course, for such a matrix to be in $SL(n, \mathbf{o})$ it is necessary that the diagonal entries lie in the group of units $\mathbf{o}^\times$ of $\mathbf{o}$). We will make choices of coordinates and of chamber in the affine building, so that the stabilizer is as indicated.

Let $V = k^n$, and let $e_1, \ldots, e_n$ be the usual k-basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \ldots \\ 0 \end{pmatrix} \quad \ldots \; e_n = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Take lattices

$$L_o = \mathbf{o}e_1 + \ldots + \mathbf{o}e_n$$
$$L_1 = \mathbf{m}^{-1}e_1 + \mathbf{o}e_2 + \ldots + \mathbf{o}e_n$$
$$L_2 = \mathbf{m}^{-1}(\mathbf{o}e_1 + \mathbf{o}e_2) + \mathbf{o}e_3 + \ldots + \mathbf{o}e_n$$
$$\ldots$$
$$L_{n-1} = \mathbf{m}^{-1}(\mathbf{o}e_1 + \ldots + \mathbf{o}e_{n-1}) + \mathbf{o}e_n$$

We obtain a periodic flag of lattices as before by taking

$$L_{i-n\ell} = \mathbf{m}^\ell L_i$$

The stabilizer of this flag of lattices is, by the construction, the subgroup B stabilizing a chamber in the affine building. To see what B is, observe first that the stabilizer of L_o is the group $SL(n,\mathbf{o})$ of all matrices in $SL(n)$ having entries in $\mathbf{o}$, using the e_i coordinates to write matrices. And this group $SL(n,\mathbf{o})$ also stabilizes $\mathbf{m}^{-1}L_o$.

All the quotients L_i/L_o for $0 < i < n$ are vectorspaces over the residue field $\kappa = \mathbf{o}/\mathbf{m}$, and are κ-subspaces of $\mathbf{m}^{-1}L_o/L_o$. The flag

$$L_1/L_o,\ \ L_2/L_o,\ \ L_3/L_o,\ \ \ldots,\ \ L_{n-1}/L_o$$

is a maximal flag of κ-subspaces of $(\mathbf{m}^{-1}L_o)/L_o$. Using the images of $\mathbf{m}e_i$ as κ-basis for this space, this flag is none other than the *standard* flag of subspaces in that vectorspace.

Observe that if a matrix $g \in SL(n,\mathbf{o})$ has entries which differ by elements of the ideal $\mathbf{m}$ from the entries of the *identity* matrix, then for $v \in \mathbf{m}^{-1}L_o$ we have

$$g(v + L_o) = (gv) + L_o$$

To see this, write $g = 1 + \mathbf{m}T$ with T a matrix having entries in $\mathbf{o}$. Then

$$g(v + L_o) = (1 + \mathbf{m}T)v + gL_o = v + \mathbf{m}Tv + L_o =$$
$$= v + \mathbf{m}T(\mathbf{m}^{-1}L_o) + L_o = v + TL_o + L_o = v + L_o$$

That is, matrices of this form act *trivially* on the quotient $(\mathbf{m}^{-1}L_o)/L_o$.

There is a little hazard here, since the chambers are defined by *homothety classes* of lattices, not just by the lattices themselves. Thus, elements $g \in SL(n,k)$ which map L_o to *any* lattice $\mathbf{m}^\ell L_o$ (in the homothety class of L_o) certainly stabilize the homothety class $[L_o]$ of L_o. But the determinant of such g would have *ord* equal to $n\ell$. For g to be in $SL(n,k)$ it must be that $\ell = 0$. Thus, after all, if $g \in SL(n,k)$ stabilizes the homothety class of a lattice, then g actually stabilizes the lattice itself.

Thus, it is clear that the Iwahori subgroup B is the collection of matrices in $SL(n,\mathbf{o})$ which modulo $\mathbf{m}$ are *upper-triangular* elements of $SL(n,\kappa)$.

19.5 The maximal apartment system

In order to apply results of (4.6) and chapter 17 which use the spherical building at infinity, it is necessary to know that $SL(V)$ acts strongly transitively with respect to the *maximal* apartment system. This is not so for *arbitrary* discrete valuation rings $\mathbf{o}$:

Theorem: If the discrete valuation ring $\mathbf{o}$ is *complete* and its fraction field k is *locally compact* then we have strong transitivity of $SL(V)$ with reference to the *maximal* apartment system in the thick affine building X constructed above.

And as noted in (17.7) the proposition has a corollary bearing upon the apartment system constructed above:

Corollary: If k is locally compact then the apartment system $\mathcal{A}$ constructed above *is the maximal one*.

Proof of corollary: By its definition, the strong transitivity implies ordinary transitivity on the collection of apartments. ♣

Remarks: The truth of this corollary is certainly not clear *a priori*, and does indeed depend upon *completeness* of the discrete valuation ring, which was in no way used up to this point.

Proof of proposition: In fact this result does not depend much upon the particulars of this situation. Rather, quite generally, if the Iwahori subgroup 'B' in an affine BN-pair is *compact and open* in a group G_o acting strongly transitively on an affine building (and preserving types), then G_o is strongly transitive with the *maximal* apartment system (17.7).

In terms of the previous section, B is the intersection of $SL(n,k)$ with the subset U of the space of n-by-n matrices described as follows. Let the $(i,j)^{\text{th}}$ entry of a matrix x be $x_{i,j}$. Then consider the conditions

$$\operatorname{ord}(x_{i,j}) > -1 \quad \text{for} \quad i \leq j$$

$$\operatorname{ord}(x_{i,j}) > 0 \quad \text{for} \quad i > j$$

where as usual ord $= n$ on $\mathbf{m}^n$. This describes U as an open set. Since $B = U \cap SL(n,k)$, this shows that B is *open*.

The open-ness of B implies that any translate Bw' of B by $w' \in G$ is open, so any union BwB of sets Bw' is open. By the Bruhat-Tits decomposition, G is a disjoint union of sets of the form BwB. Thus, the complement of B is open, so B is closed.

Then the *compactness* of the closed set B follows from the local compactness of $SL(n,k)$, which follows from the local compactness of V, which follows from the assumed local compactness of k because V is finite-dimensional. ♣

20. Construction of Affine Buildings for Isometry Groups

- Affine buildings for alternating spaces
- The double oriflamme complex
- The (affine) single oriflamme complex
- Verification of the building axioms
- Group actions on the buildings
- The maximal apartment systems

The buildings constructed here are attached to non-degenerate alternating or quadratic forms on vectorspaces over the fraction field k of a discrete valuation ring $\mathbf{o}$. For the finer results it must be assumed that the discrete valuation ring is *locally compact* with respect to the metric associated to the valuation. Just as for $SL(V)$ (19.1), the notion of $\mathbf{o}$-*lattice* (18.3) plays a central role, comparable to the role played by subspaces and isotropic subspaces in construction of spherical buildings.

In the three families of examples here, the apartments are Coxeter complexes attached to the Coxeter systems (W, S) of types $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$ (2.2). The verification that these buildings truly are *affine*, via the Coxeter data criterion (13.6), was done in (13.8), affineness would follow from checking that in each case the Coxeter data is as claimed.

The first construction (type $\tilde{C}_n$), for alternating spaces, is a synthesis of ideas from the *spherical* construction for isometry groups, together with ideas from the *affine* construction for $\tilde{A}_n$ in the last chapter. By contrast, the second family (type $\tilde{D}_n$), for quadratic spaces which are orthogonal sums of hyperbolic planes, requires use of the oriflamme trick (11.1) *twice*. The third example (type $\tilde{B}_n$), encompassing most other quadratic spaces, combines elements of both the affine $\tilde{C}_n$ and the double oriflamme complex.

After the construction, viewpoints and methods already illustrated in the spherical examples and for the affine $\tilde{A}_n$ suffice to verify that the complexes are buildings as claimed, and that the groups act strongly transitively. By this point, the detailed descriptions of the buildings suggest most of the details of this verification. Thus, by now the main point is the *construction*, after which the rest is just mopping-up.

In all these cases the Iwahori subgroup 'B', by definition the stabilizer of a chamber, has a simple description in suitable coordinates: it consists of matrices in the group which have entries in $\mathbf{o}$ and which, reduced modulo $\mathbf{m}$, lie in a minimal parabolic subgroup of the corresponding alternating or orthogonal group over the *residue field*.

19.6 Affine buildings for alternating spaces

Here we construct a *(thick) affine building* for a non-degenerate alternating space V of dimension $2n$ over the fraction field k of a discrete valuation ring $\mathbf{o}$.

As in every other example, the procedure is that we describe an *incidence geometry* from which we obtain a *flag complex* which one verifies is a *thick building* by checking the axioms. Once the Coxeter data is identified as $\tilde{C}_n$, the computations already done assure that the building is indeed *affine* (13.8).

Let $\mathbf{m}$ be the unique non-zero prime ideal $\mathbf{m}$ in $\mathbf{o}$. Let $\kappa = \mathbf{o}/\mathbf{m}$ be the residue field. Let V be given a non-degenerate alternating form $\langle,\rangle$ (7.2).

We need the notion of **primitive o-lattice** or simply *primitive lattice* for the form $\langle,\rangle$. Say that a lattice Λ inside V is **primitive** if $\langle,\rangle$ is $\mathbf{o}$-valued on $\Lambda \times \Lambda$, and so that $\langle,\rangle$-modulo-$\mathbf{m}$ is a *non-degenerate* κ-valued alternating form on the κ-vectorspace $\Lambda/\mathbf{m}\Lambda$. The existence of primitive lattices is straightforward: let $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$ be n hyperbolic pairs so that V is an *orthogonal* sum

$$V = \bigoplus (ke_i + kf_i)$$

Then

$$\Lambda = \sum_i \mathbf{o}e_i + \mathbf{o}f_i$$

is readily seen to be a primitive lattice.

The collection Ξ of vertices for our incidence geometry is the set of homothety classes $[\Lambda]$ of lattices Λ in V which possess a representative Λ with the following property: first, there must be a lattice Λ_o so that $\mathbf{m}^{-1}\Lambda_o$ is a *primitive* lattice, and so that

$$\Lambda_o \subset \Lambda \subset \mathbf{m}^{-1}\Lambda_o$$

and so that

$$\langle\Lambda, \Lambda\rangle \subseteq \mathbf{m}$$

where as usual

$$\langle\Lambda, \Lambda\rangle = \{\langle v, v'\rangle : v, v' \in \Lambda\}$$

The condition on Λ can be paraphrased in a helpful form: it demands that Λ/Λ_o be a totally isotropic κ-subspace of the κ-vectorspace $\mathbf{m}^{-1}\Lambda_o/\Lambda_o$ which has the non-degenerate κ-valued alternating form $\langle,\rangle$-mod-$\mathbf{m}$.

Define an incidence relation $\sim$ on Ξ as follows: write $\xi \sim \xi'$ for $\xi, \xi' \in \Xi$ if there are lattices $x \in \xi$ and $y \in \xi'$ and a lattice Λ_o so that $\mathbf{m}^{-1}\Lambda_o$ is primitive, so that

$$\Lambda_o \subset x \subset \mathbf{m}^{-1}\Lambda_o$$

$$\mathbf{m} \cdot \Lambda_o \subset y \subset \mathbf{m}^{-1}\Lambda_o$$

and also either $x \subset y$ or $y \subset x$.

As in general (3.1), the associated *flag complex* X is the simplicial complex with vertices Ξ and simplices which are *mutually incident* subsets of Ξ, i.e., subsets σ of Ξ so that, for all $x, y \in \sigma$, $x \sim y$.

The apartment system in X is identified as follows. First, a **frame** is an *unordered* n-tuple

$$\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$$

of *unordered* pairs $\{\lambda_i^1, \lambda_i^2\}$ of lines so that

$$V = (\lambda_1^1 + \lambda_1^2) + \ldots + (\lambda_n^1 + \lambda_n^2)$$

and so that

$$(\lambda_i^1 + \lambda_i^2) \perp (\lambda_j^1 + \lambda_j^2) \quad \text{for } i \neq j$$

and so that each $\lambda_i^1 + \lambda_i^2$ is a *hyperbolic plane* (in the sense of geometric algebra, (7.2)). (As usual, for two vector subspaces V_1, V_2, $V_1 \perp V_2$ means $\langle x, y \rangle = 0$ for all $x \in V_1$ and for all $y \in V_2$).

A vertex $[\Lambda]$ lies inside the apartment A specified by the frame

$$\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$$

if there are free $\mathbf{o}$-modules M_i^j inside λ_i^j so that

$$\Lambda = \bigoplus_{i,j} M_i^j$$

for some (hence, *every*) representative Λ in the homothety class.

The *maximal simplices* are unordered $(n+1)$-tuples

$$[\Lambda_o], [\Lambda_1], \ldots, [\Lambda_n]$$

of homothety classes of lattices with representatives Λ_i so that $\mathbf{m}^{-1}\Lambda_o$ is a *primitive* lattice, so that

$$\Lambda_o \subset \Lambda_i \subset \mathbf{m}^{-1}\Lambda_o \quad \text{for } 0 < i \leq n$$

and so that

$$\Lambda_1/\Lambda_o \subset \Lambda_2/\Lambda_o \subset \ldots \subset \Lambda_n/\Lambda_o$$

is a *maximal isotropic flag* of κ-subspaces in the alternating κ-vectorspace $\mathbf{m}^{-1}\Lambda_o/\Lambda_o$ with $\langle,\rangle$-mod-$\mathbf{m}$.

The maximal simplices are in bijection with *ascending chains* of lattices

$$\ldots \subset \Lambda_{-1} \subset \Lambda_o \subset \Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_n \subset \Lambda_{n+1} = \Lambda_{n-1}^* \subset$$

$$\ldots \subset \Lambda_{2n-1} = \Lambda_1^* \subset \Lambda_{2n} \subset \ldots$$

(indexed by integers) where for a lattice Λ

$$\Lambda^* = \{v \in V : \langle v, \lambda \rangle \in \mathbf{m}, \quad \text{for all} \ \ \lambda \in \Lambda\}$$

and where we require

- that $\mathbf{m}^{-1}\Lambda_o$ be a primitive lattice

- the *periodicity* property
$$\Lambda_{i+2n} = \mathbf{m}^{-1}\Lambda_i$$
for all indices i
-
$$\Lambda_1/\Lambda_o \subset \Lambda_2/\Lambda_o \subset \ldots \Lambda_n/\Lambda_o$$
is a maximal flag of totally-isotropic κ-spaces inside the non-degenerate alternating κ-vectorspace $\mathbf{m}^{-1}\Lambda_o/\Lambda_o$

Remarks: The definition of x^* above would be the same as taking all vectors whose *reduction mod* $\mathbf{m}$ is orthogonal to all vectors in x-mod-$\mathbf{m}$. More precisely, for *any primitive* lattice x_o such that $\mathbf{m}x_o \subset x \subset x_o$, the quotient $x^*/\mathbf{m}x_o$ is the orthogonal complement of $x/\mathbf{m}x_o$ in the non-degenerate quadratic space $x_o/\mathbf{m}x_o$.

19.7 The double oriflamme complex

The building constructed here is attached to a non-degenerate quadratic form $\langle,\rangle$ on a $2n$-dimensional vectorspace V over the fraction field k of a discrete valuation ring $\mathbf{o}$, under the further specific hypothesis that V *is an orthogonal direct sum of hyperbolic planes* (in the geometric algebra sense), and that $n \geq 4$.

As in every other example, whether spherical or affine, we describe an *incidence geometry* from which we obtain a *flag complex* which is a *building*.

One will see that the apartments are Coxeter complexes attached to the Coxeter system (W, S) of type $\tilde{D}_n$ described earlier (2.2) (for $n \geq 4$). The checking that this Coxeter system truly is *affine*, via the Coxeter data criterion (13.6), was done in (13.8), so all that needs to be checked is that the Coxeter data is as claimed.

Exactly as with the alternating case of the last section, a lattice Λ is **primitive** if $\langle,\rangle$ is $\mathbf{o}$-valued on Λ, and if $\langle,\rangle$-modulo-$\mathbf{m}$ is a *non-degenerate* κ-valued symmetric bilinear form on the κ-vectorspace $\Lambda/\mathbf{m}\Lambda$. The existence of primitive lattices is as straightforward as in the alternating space case, since as in that case V is a sum of hyperbolic planes: let $\{e_1, f_1\}$, $\{e_2, f_2\}$, $\ldots$, $\{e_n, f_n\}$ be hyperbolic pairs so that V is an *orthogonal* sum
$$V = \bigoplus (ke_i + kf_i)$$
Then
$$\Lambda = \sum_i \mathbf{o}e_i + \mathbf{o}f_i$$
is a primitive lattice.

The collection Ξ of vertices for the incidence geometry is the set of homothety classes $[\Lambda]$ of lattices Λ in V which possess a representative Λ with the following property: first, there must be a lattice Λ_o so that $\mathbf{m}^{-1}\Lambda_o$ is primitive, so that

$$\Lambda_o \subset \Lambda \subset \mathbf{m}^{-1}\Lambda_o$$

and so that

$$\langle \Lambda, \Lambda \rangle \subseteq \mathbf{m}$$

where

$$\langle \Lambda, \Lambda \rangle = \{\langle v, v' \rangle : v, v' \in \Lambda\}$$

Also, the κ-vectorspace $\Lambda/\mathbf{m}\Lambda$ must *not* be either 1-dimensional or $(n-1)$-dimensional. (This is where the restriction $n \geq 4$ enters).

The incidence relation $\sim$ on Ξ will have the same quirk as did that for the spherical oriflamme complex (11.1), by contrast to the spherical construction for all other isometry groups (10.1):

First, write $\xi \sim \xi'$ for $\xi, \xi' \in \Xi$ if there are lattices $x \in \xi$ and $y \in \xi'$ and a lattice Λ_o so that $\mathbf{m}^{-1}\Lambda_o$ is primitive, and so that

$$\Lambda_o \subset x \subset \mathbf{m}^{-1}\Lambda_o$$

$$\Lambda_o \subset y \subset \mathbf{m}^{-1}\Lambda_o$$

and also either $x \subset y$ or $y \subset x$.

And, also write $\xi \sim \xi'$ if $x = x/\mathbf{m} \cdot \Lambda_o$ and $y = y/\mathbf{m} \cdot \Lambda_o$ are both 0-dimensional or are both n-dimensional, and if all of

$$x/(x \cap y) \quad y/(x \cap y) \quad (x+y)/x \quad (x+y)/y$$

are one-dimensional κ-vectorspaces.

As defined earlier, as in general, the associated *flag complex* X is the simplicial complex with vertices Ξ and simplices which are *mutually incident* subsets of Ξ, that is, subsets σ of Ξ so that $x \sim y$ for all $x, y \in \sigma$.

The apartment system in X is identified as follows. First, a **frame** is an *unordered* n-tuple

$$\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$$

of unordered pairs $\{\lambda_i^1, \lambda_i^2\}$ of isotropic lines so that

$$V = (\lambda_1^1 + \lambda_1^2) + \ldots + (\lambda_n^1 + \lambda_n^2)$$

and so that

$$(\lambda_i^1 + \lambda_i^2) \perp (\lambda_j^1 + \lambda_j^2) \quad \text{for } i \neq j$$

and so that each $\lambda_i^1 + \lambda_i^2$ is a *hyperbolic plane* (in the sense of geometric algebra, (7.2)). As usual, for two vector subspaces V_1, V_2, $V_1 \perp V_2$ means $\langle x, y \rangle = 0$ for all $x \in V_1$ and for all $y \in V_2$.

A vertex $[\Lambda]$ lies inside the apartment specified by such a frame if there are free rank-one $\mathbf{o}$-modules M_i^j inside λ_i^j so that

$$\Lambda = \sum_{ij} M_i^j$$

for one (hence, for *all*) representatives Λ for the homothety class.

The *maximal simplices* are unordered $(n+1)$-tuples

$$[\Lambda_o^1],[\Lambda_o^2],[\Lambda_2],[\Lambda_3],\ldots,[\Lambda_{n-3}],[\Lambda_{n-2}],[\Lambda_n^1],[\Lambda_n^2]$$

of homothety classes of lattices with representatives so that $\mathbf{m}^{-1}\Lambda_o^1$ and $\mathbf{m}^{-1}\Lambda_o^2$ both are *primitive* lattices, so that

$$\Lambda_o^j \subset \Lambda_i \subset \Lambda_n^{j'} \subset \mathbf{m}^{-1}\Lambda_o \quad \text{for } 2 \le i \le n-2$$

for $j,j' \in \{1,2\}$, so that

$$\Lambda_i/\Lambda_o^j$$

is a j-dimensional totally isotropic κ-subspaces in the κ-vectorspace $\mathbf{m}^{-1}\Lambda_o^j/\Lambda_o^j$ with $\langle,\rangle$-mod-$\mathbf{m}$ for $2 \le i \le n-2$, and so that

$$\Lambda_n^{j'}/\Lambda_o^j$$

is an n-dimensional totally isotropic κ-subspaces in the κ-vectorspace $\mathbf{m}^{-1}\Lambda_o^j/\Lambda_o^j$ with $\langle,\rangle$-mod-$\mathbf{m}$. (Note that, indeed, the indices 1 and $n-1$ are suppressed, while the indices 0 and n are 'doubled').

Remarks: As in the case of the spherical oriflamme complex constructed for such quadratic spaces, the peculiar details are necessary to arrange that the building be *thick*.

In a manner just slightly more complicated than that for alternating spaces, the maximal simplices are in bijection with certain more-or-less *'periodic'* infinite families of lattices, as follows.

We consider infinite families of lattices in V

$$\ldots \subset \Lambda_o^1, \Lambda_o^2 \subset \Lambda_2 \subset \Lambda_3 \subset \ldots \subset \Lambda_{n-2} \subset \Lambda_n^1, \Lambda_n^2 \subset \Lambda_{n+2} = \Lambda_{n-2}^* \subset$$
$$\ldots \subset \Lambda_{2n-2} = \Lambda_2^* \subset \Lambda_{2n}^1 = \mathbf{m}^{-1}\Lambda_o^1, \Lambda_{2n}^2 = \mathbf{m}^{-1}\Lambda_o^2 \subset \ldots$$

with some further conditions. We require also the *periodicity* conditions

$$\Lambda_{i+2n\ell} = \mathbf{m}^{-\ell}\Lambda_i \quad (2 \le i \le n-2 \quad \text{or} \quad n+2 \le 2n-2$$
$$\Lambda_{2n\ell}^j = \mathbf{m}^{-\ell}\Lambda_o^j \qquad \Lambda_{n+2n\ell}^j = \mathbf{m}^{-\ell}\Lambda_n^j$$

for $j=1,2$ and for all $\ell \in \mathbb{Z}$. And we require

$$\Lambda_{2n-i} = \Lambda_i^* \quad (2 \le i \le n-2)$$

where for a lattice x we use notation

$$x^* = \{v \in V : \langle x,v\rangle \in \mathbf{m}\}$$

as was used in the case of alternating spaces.

That is, we have an infinite $2n$-periodic chain of lattices with the $n+2n\ell^{rmth}$ and $2n\ell^{\text{th}}$ items *doubled*, and the $1+2n\ell^{\text{th}}$, $(n-1)+2n\ell^{\text{th}}$, $(n+1)+2n\ell^{\text{th}}$, and $(2n-1)+2n\ell^{\text{th}}$ items *suppressed*, with additional conditions as above.

19.8 The (affine) single oriflamme complex

The building constructed here is attached to a non-degenerate quadratic form $\langle,\rangle$ on a vectorspace V over the fraction field k of a discrete valuation ring $\mathbf{o}$. We suppose that V is the orthogonal direct sum of n hyperbolic planes and an anisotropic subspace of some *positive* (but otherwise unspecified) dimension. (By Witt's theorem (7.3) the isometry class of such anisotropic summand is uniquely determined).

This pointedly excludes the special case, just treated, in which the quadratic space V is an orthogonal direct sum of hyperbolic planes. On the other hand, we must now *postulate* the existence of a *primitive lattice*, unlike the cases of alternating spaces and 'hyperbolic' quadratic spaces just treated.

The apartments are Coxeter complexes attached to the Coxeter system (W, S) of type $\tilde{B}_n$ (2.2). The *affineness* was verified in (13.8) via the Coxeter data criterion (13.6), so all we need to do here is to check that the Coxeter data is as claimed.

A **primitive lattice** (if one exists) is a lattice Λ in V so that $\langle,\rangle$ is $\mathbf{o}$-valued on Λ, and so that $\langle,\rangle$-mod-$\mathbf{m}$ is a non-degenerate κ-valued quadratic form on the κ-vectorspace $\Lambda/\mathbf{m}\Lambda$, where $\kappa = \mathbf{o}/\mathbf{m}$ is the residue field.

We assume that primitive lattices exist. In the most important examples this can be verified directly. For example, the single most important family is the following. Consider a quadratic form given in coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ by

$$x_1 y_1 + \ldots + x_n y_n + \eta z^2$$

with η a unit in $\mathbf{o}$. In this example, the set of points where all the x_i, y_i, and z are in $\mathbf{o}$ is certainly a lattice, and is *primitive*.

The other case of general importance is the following. Consider a quadratic form given in coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z, w)$ by

$$x_1 y_1 + \ldots + x_n y_n + z^2 - \eta w^2$$

with η a non-square unit in $\mathbf{o}$. The set of points where all the x_i, y_i, and z, w are in $\mathbf{o}$ is a lattice, and is *primitive*.

These two examples cover almost all the situations that occur in practice.

The vertices Ξ of the incidence geometry are homothety classes $[\Lambda]$ of lattices with representatives Λ so that there is a *primitive* lattice Λ_o so that

$$\mathbf{m}\Lambda_o \subset \Lambda \subset \Lambda_o$$

and so that

$$\Lambda/\mathbf{m}\Lambda_o$$

is a totally isotropic κ-subspace of the non-degenerate quadratic κ-space

$$\Lambda_o/\mathbf{m}\Lambda_o$$

(with κ-valued quadratic form $\langle , \rangle$-mod-$\mathbf{m}$). And we require that the κ-dimension of $\Lambda/\mathbf{m}\Lambda_o$ *not be* 1.

Define an incidence relation $\sim$ on Ξ as follows: two vertices ξ and ξ' can be incident in *two* ways. *First*, $\xi \sim \xi'$ if there are lattices $x \in \xi$ and $y \in \xi'$ and a lattice Λ_o so that $\mathbf{m}^{-1}\Lambda_o$ is primitive, and

$$\Lambda_o \subset x \subset \mathbf{m}^{-1}\Lambda_o$$
$$\Lambda_o \subset y \subset \mathbf{m}^{-1}\Lambda_o$$

and also either $x \subset y$ or $y \subset x$. *Second*, $\xi \sim \xi'$ if there are representatives $x \in \xi$, $y \in \xi'$ both of which are primitive lattices, and if all the quotients

$$x/(x \cap y) \quad y/(x \cap y) \quad (x+y)/x \quad (x+y)/y$$

are one-dimensional κ-spaces.

As defined earlier in general (3.1), the associated *flag complex* X is the simplicial complex with vertices Ξ and simplices which are *mutually incident* subsets of Ξ, i.e., subsets σ of Ξ so that, for all $x, y \in \sigma$, $x \sim y$.

The apartment system in X is identified as follows. First, a **frame** is an *unordered* n-tuple

$$\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$$

of unordered pairs λ_i^1, λ_i^2 of isotropic lines so that $H_i = \lambda_i^1 + \lambda_i^2$ is a hyperbolic plane, and so that the hyperbolic planes spaces H_i are mutually orthogonal. A vertex $[\Lambda]$ lies inside the apartment given by such a frame if there are rank-one $\mathbf{o}$-modules M_i^j in λ_i^j so that

$$\Lambda = (\bigoplus_{i\,j} M_i^j) \oplus \Lambda^+$$

where Λ^+ is the unique maximal $\mathbf{o}$-lattice on which $\langle , \rangle$ is $\mathbf{o}$-valued (18.3), inside the anisotropic quadratic k-vectorspace

$$(\bigoplus_{i\,j} \lambda_i^j)^\perp$$

(We invoke Witt's theorem (7.3) to know that this orthogonal complement is anisotropic).

The following lemma is necessary in order to be sure of adequate uniqueness for lattices in anisotropic spaces. In the important examples where the V is a sum of hyperbolic planes and an additional one-dimensional space, a much more elementary proof can be given. For more general purposes, however, it seems that no very much weaker hypothesis than that the field k is a (non-trivial) *complete discretely-valued (ultrametric) field* (18.3) will suffice. For simplicity, we suppose that 2 is a unit in $\mathbf{o}$.

Lemma: Let V^+ be an *anisotropic* quadratic space over k. Suppose that k is a *complete discretely-valued field*. Suppose that 2 is a unit in the valuation ring $\mathbf{o}$. Then there is a *unique* maximal lattice on which $\langle , \rangle$ has values in the valuation ring $\mathbf{o}$.

Proof: More specifically, we claim that

$$\Lambda^+ = \{v \in V^+ : \langle v, v\rangle \in \mathbf{o}\}$$

is the unique maximal lattice as described in the statement of the lemma. The issue is verification that Λ^+ is closed under sums.

Suppose that there are x, y in Λ^+ so that $z = x + y$ is not contained in Λ^+, and reach a contradiction to the condition of anisotropy.

Certainly we may suppose that x and y are *primitive* in Λ^+, meaning that neither $\varpi^{-1}x$ nor $\varpi^{-1}y$ are in Λ^+. This entails that $\langle \varpi^{-1}x, \varpi^{-1}x\rangle$ is not in $\mathbf{o}$, while $\langle x, x\rangle$ itself *is* in $\mathbf{o}$. Thus,

$$\operatorname{ord}\langle x, x\rangle = 0 \quad \text{or} \quad 1$$

and similarly for y. On the other hand, since z is not in Λ^+, and since

$$\langle z, z\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$$

we conclude that

$$\operatorname{ord}\langle x, y\rangle \leq -1$$

(and using the hypothesis that 2 is a unit).

By symmetry, we may suppose that $\operatorname{ord}\langle y, y\rangle \geq \operatorname{ord}\langle x, x\rangle$. Let ϖ^n be the smallest power of the local parameter ϖ so that

$$\frac{\langle x, \varpi^n y\rangle}{\langle x, x\rangle} \in \mathbf{o}$$

and define

$$f(\alpha) = \langle \alpha x + \varpi^n y, \alpha x + \varpi^n y\langle \cdot \rangle x, x\rangle$$

Rearranging, this is

$$f(\alpha) = \alpha^2 + 2\alpha \cdot \frac{\langle x, \varpi^n y\rangle}{\langle x, x\rangle} + \frac{\langle \varpi^n y, \varpi^n y\rangle}{\langle x, x\rangle}$$

By the choice of n (and the assumption that 2 is a unit), the coefficient of the linear term is a unit. Since z was assumed *not* to lie in Λ^+, it must be that $n > 0$, so the constant term has *ord* strictly positive.

Thus, by Hensel's Lemma (18.3), there is a root in k of $f(\alpha) = 0$. But this would imply that there is an α so that

$$\langle \alpha x + \varpi^n y, \alpha x + \varpi^n y\langle = 0$$

which would contradict the assumption of anisotropy. Thus, it must have been that $x + y$ lay in Λ^+ after all. ♣

The *maximal simplices* are unordered $(n + 1)$-tuples

$$[\Lambda_o^1], [\Lambda_o^2], [\Lambda_2], [\Lambda_3], \ldots, [\Lambda_{n-1}], [\Lambda_n]$$

of homothety classes of lattices with representatives $\Lambda_o^1, \Lambda_o^2, \Lambda_2, \Lambda_3, \ldots, \Lambda_n$ so that $\mathbf{m}^{-1}\Lambda_o^1$ and $\mathbf{m}^{-1}\Lambda_o^2$ are *primitive* lattices, so that for both $j = 1, 2$

$$\Lambda_o^j \subset \Lambda_i \subset \mathbf{m}^{-1}\Lambda_o^j \quad \text{for } 1 < i \leq n$$

so that

$$\Lambda_2/\Lambda_o^j \subset \Lambda_3/\Lambda_o^j \subset \ldots \subset \Lambda_n/\Lambda_o^j$$

in the κ-vectorspace $\mathbf{m}^{-1}\Lambda_o^j/\Lambda_o^j$ and so that Λ_i/Λ_o^j is an i-dimensional totally isotropic subspace of $\mathbf{m}^{-1}\Lambda_o^j/\Lambda_o^j$ (with $\langle,\rangle$-mod-$\mathbf{m}$).

Remarks: As in the case of the spherical oriflamme complex constructed for quadratic spaces which are orthogonal sums of hyperbolic planes, the peculiar details are necessary to arrange that the building be *thick*.

We consider infinite families of lattices in V

$$\ldots \subset \Lambda_o^1, \Lambda_o^2 \subset \Lambda_2 \subset \Lambda_3 \subset \ldots \subset \Lambda_{n-1} \subset \Lambda_n \subset \Lambda_{n+1} = \Lambda_{n-1}^* \subset$$

$$\ldots \subset \Lambda_{2n-2} = \Lambda_2^* \subset \Lambda_{2n}^1 = \mathbf{m}^{-1}\Lambda_o^1, \Lambda_{2n}^2 = \mathbf{m}^{-1}\Lambda_o^2 \subset \ldots$$

with some further conditions. We require also the *periodicity* conditions

$$\Lambda_{i+2n\ell} = \mathbf{m}^{-\ell}\Lambda_i \quad (2 \leq i \leq 2n-2 \quad \text{and} \quad \ell \in \mathbb{Z})$$

$$\Lambda_{2n\ell}^j = \mathbf{m}^{-\ell}\Lambda_o^j$$

for $j = 1, 2$ and for all $\ell \in \mathbb{Z}$. And for a lattice x we use the notation

$$x^* = \{v \in V : \langle x, v\rangle \in \mathbf{m}\}$$

as in the previous two sections.

That is, we have an infinite $2n$-periodic chain of lattices with the $2n\ell^{\text{th}}$ items *doubled*, and the $1 + 2n\ell^{\text{th}}$ and $(2n-1) + 2n\ell^{\text{th}}$ items *suppressed*, and with additional conditions.

19.9 Verification of the building axioms

Methods already illustrated suffice to prove that the three families of constructions just above yield thick affine buildings. *Keep the notation of the previous three sections.*

As in all previous examples, the program of the proof is:

- Granting that the apartments are chamber complexes, show that they are *thin*
- Granting that the whole thing is a chamber complex, show it is *thick*
- Show that the apartments are chamber complexes, by studying reflections across facets
- Show that any two maximal simplices lie inside a common apartment (thereby also showing that the whole is a chamber complex, in light of the previous point)
- Show that two apartments with a common chamber are isomorphic, by a simplicial isomorphism fixing their intersection pointwise.

- Determine Coxeter data by reviewing reflections across facets

In all three examples, we think of a maximal simplex as being a more-or-less *infinite periodic chain of lattices* (with additional properties varying among the three examples), as done above, and refer to that viewpoint throughout. The oriflamme trick, which amounts to *suppressing* an index while *doubling* an adjacent index, should be viewed as a technical modification of the basic idea of using infinite periodic lattices.

Indeed, the spherical oriflamme construction (11.1) was merely a variant upon the idea of using flags of isotropic subspaces, as employed for isometry groups in general (10.1).

We address the indicated issues roughly in order, taking advantage of the details worked out in previous examples. A little terminology is handy: in the double oriflamme construction, we would say that 0 and n are **doubled** indices, while 1 and $n-1$ are **suppressed** indices (reflecting the oriflamme construction's deviation from the analogue for alternating spaces). In the case of alternating spaces, there are *no* doubled and *no* suppressed indices. In the case of the affine single oriflamme complex, 1 is *suppressed*, while 0 is *doubled*.

First, consider the other chambers with facet F_i, the latter obtained by omitting the i^{th} lattice Λ_i, where

- $0 \leq i-1 < i+1 \leq n$
- None of $i-1$, i, $i+1$ is *suppressed* or *doubled*.

Finding other chambers with this facet amounts to choice of another κ vector subspace of $\Lambda_{i-1}/\Lambda_{i+1}$. That is, the issue here is identical to the analogous issue for the affine building for lattices, treated in chapter 19. For that matter, that issue itself really was equivalent to the analogous issue for the *spherical* building for (unadorned) vectorspaces over the residue field κ.

Thus, by computations we've already done, in a fixed apartment, there are only two choices for such intermediate space, depending upon the two lines (in the frame) along which Λ_{i-1} differs from Λ_{i+1}. And in the whole building, the choice of a one-dimensional subspace in a two-dimensional space offers at least

$$(q^2-1)/(q-1) = q+1 \geq 2+1 = 3$$

choices even for κ finite with q elements.

Now consider the case that $i-1=0$, so $i-1$ is a *suppressed* index. Then $i-2=0$ is *doubled*, and in fact choice of another chamber Λ_i' with facet F_i corresponds to choice of another one-dimensional κ vector subspace inside

$$\Lambda_2/(\Lambda_o^1+\Lambda_o^2)$$

which itself is two-dimensional over κ. Thus, the thin-ness and thick-ness hold. Similarly, if $i+1=n$, so that $i+1$ is suppressed, then we look at other

κ-lines in

$$(\Lambda_n^1 \cap \Lambda_n^2)/\Lambda_{n-3}$$

The two tricks are combined if both $i \pm 1$ are *suppressed* (in the $n = 4$ case for the double oriflamme complex).

Next, consider $i = n$ in the alternating case. The choice of another chamber with facet F_i is equivalent to choice of another κ-line inside the two-dimensional κ-space $\Lambda_{n-1}^*/\Lambda_{n-1}$. That is, the issue reverts to the analogue in the *spherical* building for an alternating space (10.2). Thus, in a fixed apartment, there are altogether two choices, while in the whole building there are $q+1$ for a field κ of cardinality q.

And, as it happens, the case of $i = 0$ in the alternating-space case is nearly identical to the $i = n$.

Consider the facet F_n^1 corresponding to dropping Λ_n^j in the double oriflamme case. (The case of F_n^2 is of course completely symmetrical). This is nearly identical to the spherical oriflamme case. The choice of another chamber with this facet is equivalent to choice of a totally isotropic κ subspace of the four-dimensional space $\Lambda_{n-2}^*/\Lambda_{n-2}$ whose intersection with $\Lambda_n^2/\Lambda_{n-2}$ is one-dimensional. As in the case of the spherical oriflamme complex (11.1), within a specified apartment there are only two possibilities (including the original), while in the whole building there are at least three.

Less obvious is the case of F_o^1 (and F_o^2) in both the single and double oriflamme complexes. But in fact the argument is a minor variant of the F_n^1 and F_o^2 discussion (which is essentially identical to the spherical oriflamme case (11.1)).

Thus, in all three families, granting that the apartments are chamber complexes, they are *thin*; and, granting that the building is a chamber complex, it is *thick*

Next, we will see that each apartment is a chamber complex: there is a gallery from any maximal simplex to any other maximal simplex in the same apartment. To see this, we consider the effect of moving (inside the given apartment) across the facets of maximal simplices.

First, we consider the chambers in a fixed apartment having a common vertex $[x]$, with $x = \Lambda_o$ in the alternating space case, and $x = \Lambda_o^1$ in the quadratic-space case. Thus, $\mathbf{m}^{-1}$ is a primitive lattice. Looking at flags of *lattices* modulo x converts the question into one about flags of *vectorspaces* over the residue field κ. The latter question is exactly that already treated in discussion of *spherical* buildings in chapters 10 and 11. That is, we have *already shown* that the movements across facets with vertex x connect all the chambers with vertex $[x]$ (in a fixed apartment) by galleries.

Next, consider two vertices $[x]$ and $[x']$ in a fixed apartment, with x and x' *primitive* lattices. In light of the previous paragraph, to prove that the apartments really are chamber complexes, it would suffice to show that *some* maximal simplex with vertex $[x]$ is connected by a gallery to *some* maximal simplex with vertex $[x']$. In all three examples, this follows by a straightforward adaptation of the analogous argument used for $\tilde{A}_n$ (19.2).

This completes the outline of the argument that apartments are chamber complexes.

Next, we see that any two maximal simplices lie in a common apartment. This is one of the building axioms, and in light of the previous bit of discussion, upon completion of this item we will also know then that the whole complex is a chamber complex.

Since all three families of examples have apartments specified by *frames*, meaning certain families of isotropic lines occuring in pairs, the goal would be to find a common frame to fit two given maximal simplices. This is made slightly more complicated by the quite real possibility that there be *more than one* apartment containing the two chambers, so that there is no unique characterization of 'the' common apartment.

Rather, we turn again to the description of maximal simplices in terms of infinite periodic chains of lattices, and compare two such via a Jordan-Holder-type argument. In the case of the spherical A_n and the affine $\tilde{A}_n$, the argument was literally that of Jordan-Holder, while in the cases of spherical C_n and the spherical oriflamme, geometric algebra was used to more sharply describe the comparison. Either of these approaches succeeds here, and we will not repeat them further. Thus, we grant ourselves this building axiom, and also grant that the building is a chamber complex.

Now we consider the other building axiom: given a simplex x and a chamber C both lying in two apartments A, B, show that there is an isomorphism $B \to A$ fixing both x and C pointwise. As in all other examples, it turns out to be simpler to prove the apparently stronger assertion that, given two apartments A, B containing a chamber C, there is an isomorphism $f : B \to A$ fixing $A \cap B$ pointwise.

As in all earlier examples, in these three families of affine buildings there is a *unique* isomorphism $f : B \to A$ describable in terms of the defining frames and fixing C pointwise, and this unique isomorphism is readily proven to fix all of $A \cap B$.

Finally, we consider the Coxeter data.

In the case of $\tilde{C}_n$, the computations in the spherical case C_n (10.3) determine all the Coxeter data except those bits regarding the reflection s_o through the facet F_o obtained by omitting the 0^{th} lattice in a flag. But, in fact, the interaction of s_o with the reflection s_1 through F_1 is identical to the interaction of the reflection through F_n with the reflection through F_{n-1}, and and commutes with all others. Thus, the $\tilde{C}_n$ system is obtained from the C_n system by adjoining another reflection s_o, with $m(s_o, s_1) = 4$ and $m(s_o, s_i) = 2$ for $i > 1$ (where the indices are arranged so that also $m(s_n, s_{n-1}) = 4$).

In the case of the affine single oriflamme complex $\tilde{B}_n$ and the double oriflamme complex $\tilde{D}_n$, most of the Coxeter data is determined just as in the spherical case (10.2), (11.2). This is true of the oriflamme-doubling of the 0^{th} index in both cases, in addition to the previously-considered doubling of index already present in the spherical oriflamme complex (11.2). Thus, we have reflections $s_o^{(1)}$, $s_o^{(2)}$, s_2, $s_3, \ldots$, suppressing the index 1 in both cases. And $m(s_o^{(i)}, s_2) = 3$, while both $s_o^{(1)}$ and $s_o^{(2)}$ commute with everything else (including each other). The rest of the relations are identical to the spherical cases C_n and D_n for $\tilde{B}_n$ and $\tilde{D}_n$, respectively.

These remarks should be a sufficient indication of all the proofs, which can be almost entirely reconstituted from previous arguments.

19.10 Group actions on the buildings

Keep the previous notation used in this chapter.

In some slightly mysterious way, most of the labor in the larger story of construction of a building and examination of a group action upon it goes into being sure that the building is as claimed, after which the requisite properties of the group action are most often relatively easy to check. In particular, in all the examples we have considered, all that is needed is a sufficient supply of *monomial matrices*, meaning that in suitable coordinates there is just one non-zero matrix entry in each row and column. The 'suitable coordinates' invariably refer to a maximal orthogonal direct sum of hyperbolic planes inside the space, ignoring whatever anisotropic orthogonal complement (if any) remains afterward.

It is also slightly mysterious, but as well fortuitous, that showing that the stabilizer of an apartment is transitive on chambers within it is always easy. By contrast, showing directly that the stabilizer of a chamber is transitive on apartments containing it appears to be non-trivial. As it is, the (label-preserving) stabilizer of an apartment always is essentially a group of monomial matrices, in coordinates which refer to a maximal family of mutually orthogonal hyperbolic planes.

In the case of (non-degenerate) alternating spaces V, every such space is an orthogonal direct sum of hyperbolic planes. Thus, the only invariant is *dimension*, which must be *even*. If V is of dimension $2n$, the corresponding isometry group (symplectic group) $Sp(V)$ is often denoted simply $Sp(n)$ (or, in some circles, $Sp(2n)$). The tangible labelling on the associated affine building should be constructed in the same manner as that for $SL(V)$: fix one vertex $[\Lambda_o]$ with Λ_o a *primitive* lattice, and for any other class $[\Lambda]$ choose a representative Λ so that $\mathbf{m}\Lambda_o \subset \Lambda \subset \Lambda_o$, and let the *type* of Λ be the ℓ-mod-$2n$ where Λ/Λ_o is a κ-vectorspace of dimension ℓ.

Then, since always $Sp(V) \subset SL(V)$, unavoidably this symplectic group preserves labels.

Witt's theorem assures that any two frames (specified by n-tuples of pairs of lines, pairwise forming hyperbolic planes, etc., as above) can be mapped to each other by an isometry. This is the transitivity of the group on apartments (specified by frames).

Using the coordinates from the isotropic lines making up a given frame, it is immediate that the stabilizer of the corresponding apartment consists of all isometries whose matrix has exactly one non-zero entry in each row and column. (These are the so-called *monomial* matrices).

To prove (in the alternating space case) that apartment stabilizers are transitive on chambers within the apartment, we use the description of chambers in terms of periodic infinite chains of lattices (with some further conditions (20.1)). Indeed, we further paraphrase this description, as follows.

Let the frame specifying the apartment be

$$\{\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}\}$$

This is an unordered n-tuple of unordered pairs of lines, so that the sum $H_i = \lambda_i^1 + \lambda_i^2$ of each pair of lines is a hyperbolic plane, and so that the hyperbolic planes H_i are mutually orthogonal. Some notation is necessary: for $\varepsilon \in \{1, 2\}$, let ε' be the *other* of the two elements of the set $\{1, 2\}$. Fix an *ordering*

$$H_{i_1}, H_{i_2}, \ldots, H_{i_n}$$

of the hyperbolic planes, together with a *choice of line* $\lambda_i^{\varepsilon_i}$ from among $\{\lambda_i^1, \lambda_i^2\}$, and a *choice of rank one lattice* M_{i_j} inside $\lambda_{i_j}^{\varepsilon_j}$. Put

$$M'_{i_j} = \{v \in \lambda_{i_j}^{\varepsilon'_j} : \langle v, w\rangle \in \mathbf{o} \quad \text{for all} \quad w \in M_{i_j}\}$$

Then put

$$\Lambda_o = M_{i_1} + M_{i_2} + \ldots + M_{i_n} + M'_{i_{n-1}} + M'_{i_{n-2}} + \ldots + M'_{i_1}$$

Thus, by construction, this Λ_o is a primitive lattice. Generally, for $0 \le j \le n$, put

$$\begin{aligned} \Lambda_j \quad = \quad & \mathbf{m}^{-1}(M_{i_1} + M_{i_2} + \ldots + M_{i_{j-1}}) \\ & + M_{i_j} + \ldots + M_{i_n} + M'_{i_{n-1}} + M'_{i_{n-2}} + \ldots + M'_{i_1} \end{aligned}$$

The extend the chain of lattices

$$\Lambda_o \subset \ldots \Lambda_n$$

first by the usual condition

$$\Lambda_{2n-j} = \Lambda_j^*$$

and then by the periodicity condition

$$\Lambda_{j+2n\ell} = \mathbf{m}^{-\ell}\Lambda_j$$

where as above

$$\Lambda^* = \{v \in V : \langle v, x\rangle \in \mathbf{m} \quad \text{for all} \quad x \in \Lambda\}$$

Each of the n choices of a line $\lambda_i^{\varepsilon_i}$ can be reversed by use of a monomial matrix inside the isometry group. And choice of rank one module inside $\lambda_i^{\varepsilon_i}$ can be altered by a monomial matrix inside the isometry group (simulataneous with adjustment of the corresonding module inside $\lambda_i^{\varepsilon_i'}$).

This is the desired transitivity, giving the strong transitivity in the case of symplectic groups.

The issues for both double oriflamme and single oriflamme complex are nearly identical to the above, except for the slight increase in notational complexity due to the suppression and doubling of indices, just as with the spherical oriflamme (11.3).

With regard to the latter, there is one significant complication, just as in the spherical oriflamme case (11.3): the modification of the labeling necessitated by the oriflamme trick causes the whole orthogonal (isometry) group *not* to preserve labels. Rather, the label-preserving group inside the isometry group is only the *special* orthogonal group, consisting of isometries with determinant 1.

19.11 Iwahori subgroups

In this section we choose convenient coordinate systems in which to describe the Iwahori subgroups (pointwise fixers of chambers) in our three examples. In all these cases, in suitable coordinates, the Iwahori subgroup 'B' consists of matrices which have entries in $\mathbf{o}$ and which, reduced modulo $\mathbf{m}$, lie in a minimal parabolic subgroup of the corresponding alternating or orthogonal group over the *residue field*.

In the first place, in each of the three families under consideration, the (label-preserving) stabilizer of a chamber must fix all the vertices of the chamber, which are homothety classes of lattices. So the Iwahori subgroup associated to the chamber is contained in the subgroup fixing the homothety class of some lattice Λ.

Let $e_1, \ldots, e_N$ be an $\mathbf{o}$-basis for Λ. Then, for an isometry g of V fixing the homothety class $[\Lambda]$, let $\alpha \in k^\times$ be so that

$$g\Lambda = \alpha \cdot \Lambda$$

Since g is an isometry, it must be that

$$\{\langle v, w\rangle : v, w \in \Lambda\} = \{\langle v, w\rangle : v, w \in \alpha \cdot \Lambda\} = \alpha^2\{\langle v, w\rangle : v, w \in \Lambda\}$$

Thus, α must be a *unit*, and not only is the homethety class preserved, but in fact that lattice itself:

$$g\Lambda = \Lambda$$

Then

$$ge_i = \sum_j ; c_{ij} e_j$$

with $c_{ij} \in \mathbf{o}$. On the other hand, g^{-1} has the same property, since $g\Lambda = \Lambda$ rather than merely $g\Lambda \subset \Lambda$. Thus, $\det g$ is necessarily a unit. That is, the matrix for such g has entires in $\mathbf{o}$ and has determinant in the units of the ring $\mathbf{o}$.

Now suppose, as occurs in the three constructions, that g fixes a *primitive* lattice $\mathbf{m}^{-1}\Lambda_o$. In the alternating space case, such a chamber fixes an infinite periodic chain of lattices

$$\ldots \subset \Lambda_o \subset \Lambda_1 \subset \ldots$$

and the chain

$$\Lambda_1/\Lambda_o \subset \Lambda_2/\Lambda_o \subset \ldots$$

is a maximal flag of totally isotropic κ-subspaces in the non-degenerate κ-vectorspace $\mathbf{m}^{-1}\Lambda_o/\Lambda_o$ with the form $\langle,\rangle$-mod-$\mathbf{m}$. Thus, with suitable choice of $\mathbf{o}$-basis for Λ_o, modulo $\mathbf{m}$ the matrices in the Iwahori subgroup are in the minimal parabolic subgroup attached to this maximal flag modulo κ.

For the double oriflamme complex, with *two* primitive lattices Λ_o^1 and Λ_o^2, the configuration of totally isotropic subspaces fixed by an element of an Iwahori subgroup is of the form

$$(\Lambda_o^1 + \Lambda_o^2)/\Lambda_o^1 \subset \Lambda_2/\Lambda_o^1 \subset \ldots \subset \Lambda_{n-2}/\Lambda_o^1 \subset \ldots \subset \Lambda_{n-2} \subset \Lambda_n^1/\Lambda_o^1,\, \Lambda_n^2/\Lambda_o^1$$

This is the same as the configuration for the spherical oriflamme complex (11.1), over the residue field κ. Note that we had to create a κ-one-dimensional isotropic subspace

$$(\Lambda_o^1 + \Lambda_o^2)/\Lambda_o^1$$

in order to match not only the *content*, but the *form* of the description.

The issue is essentially identical for the single oriflamme complex.

19.12 The maximal apartment systems

To be sure that the earlier study of the interaction of the affine building and the spherical building at infinity is applicable in the present settings, we must be sure that the apartment systems here are the *maximal* ones.

Quite generally, when the Iwahori subgroup (stabilizer of a chamber in the affine building) is *compact and open*, the apartment system is the (unique (4.4)) maximal one (17.7). To prove that the Iwahori subgroup is compact and open, we assume that the discrete valuation ring $\mathbf{o}$ is *locally compact*.

In each of our three families of examples, as was noted in the last section, in suitable coordinates the Iwahori subgroup consists of matrices in the group which have entries in $\mathbf{o}$, and which modulo $\mathbf{m}$ lie in the minimal parabolic of the corresponding isometry group over the residue field κ.

Thus, as with $SL(V)$ (18.4), (19.4), (19.5), *local compactness of the field k assures that the Iwahori subgroup is compact and open.* This assures that *the apartment systems constructed above are the maximal apartment systems.*

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